D-MODULES: AN INTRODUCTION

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1. OVERVIEW

D-modules are a useful tool in both representation theory and algebraic geometry. In this talk, I will motivate the study of D-modules by describing two similar theorems. The first is a classical result of Serre which allows for translation between algebraic geometry and commutative algebra. The second theorem of Beilinson and Bernstein is a generalization of the first which allows us to translate between representation theory and geometry. Along the way we will define D-modules and explain some of their basic properties. This talk is an overview and will not include technical details. For details and proofs, see Dragan Milicic's *Lectures on Algebraic Theory of D-Modules* (available on his website www.math.utah.edu/∼milicic).

2. MOTIVATION

Let X be an affine variety over $\mathbb{C}$. Let $R(X)$ be the ring of regular functions on X and $\mathcal{O}_X$ be the structure sheaf of X. We have that $R(X) = \Gamma(X, \mathcal{O}_X)$. Let $\mathcal{M}(R(X))$ be the category of $R(X)$-modules.

Definition 2.1. A $\mathcal{O}_X$-module $\mathcal{V}$ is *quasicoherent* if for any $x \in X$ there exists an open set U containing x such that

$$\mathcal{O}_U^{(J)} \to \mathcal{O}_U^{(I)} \to \mathcal{V}|_U \to 0$$

is exact. Here I and J are infinite indexing sets and $\mathcal{O}^{(I)} = \sum_{i \in I} \mathcal{O}$.

Let $\mathcal{M}_{qc}(\mathcal{O}_X)$ be the category of quasicoherent $\mathcal{O}_X$-modules.

Theorem 2.2. *(Serre) There is an equivalence of categories*

$$\mathcal{M}(R(X)) \longleftrightarrow \mathcal{M}_{qc}(\mathcal{O}_X)$$

This fundamental result allows us to study problems in algebraic geometry using methods of commutative algebra. The proof of this theorem relies on the following two lemmas.

Lemma 2.3. *(Serre) Let $\mathcal{V}$ be a quasicoherent $\mathcal{O}_X$-module. Then $H^p(X, \mathcal{V}) = 0$ for $p > 0$. Therefore, the functor*

$$\Gamma : \mathcal{M}_{qc}(\mathcal{O}_X) \longrightarrow \mathcal{M}(R(X))$$

is exact.

Lemma 2.4. *(Serre) Any quasicoherent $\mathcal{O}_X$-module is generated by its global sections.*

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With these two lemmas, the proof of Serre's theorem is relatively straightforward.

Proof. Let $M \in \mathcal{M}(R(X))$ and $\mathcal{V} \in \mathcal{M}_{qc}(\mathcal{O}_X)$. We have an exact sequence

$$R(X)^{(J)} \longrightarrow R(X)^{(I)} \longrightarrow M \longrightarrow 0$$

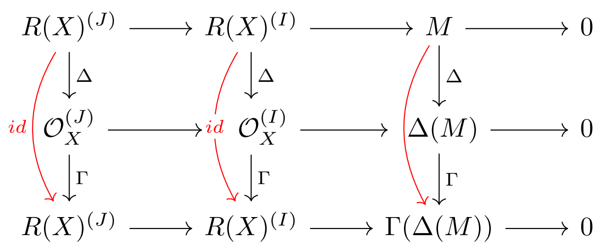
Let $\Delta(M) = \mathcal{O}_X \otimes_{R(X)} M$. This defines a right exact functor

$$\Delta \colon \mathcal{M}(R(X)) \longrightarrow \mathcal{M}_{qc}(\mathcal{O}_X)$$

We have $\Delta(R(X)^{(I)}) = \mathcal{O}_X^{(I)}$. Therefore, we can apply Δ to our exact sequence above and get the exact sequence

$$\mathcal{O}_X^{(J)} \longrightarrow \mathcal{O}_X^{(I)} \longrightarrow \Delta(M) \longrightarrow 0$$

This gives us the following commutative diagram:



Remark 2.5. For $\mathcal{V} \in \mathcal{M}_{qc}(\mathcal{O}_X)$, we have

$$\Gamma(X, \mathcal{V}) = \operatorname{Hom}_{\mathcal{O}_X}(\mathcal{O}_X, \mathcal{V}).$$

This isomorphism is given by evaluation of $\phi \in \operatorname{Hom}_{\mathcal{O}_X}(\mathcal{O}_X, \mathcal{V})$ on $1 \in \Gamma(X, \mathcal{O}_X) = R(X)$.

From the remark, we have

$$\begin{aligned} \operatorname{Hom}_{R(X)}(M, \Gamma(X, \mathcal{V})) &= \operatorname{Hom}_{R(X)}(M, \operatorname{Hom}_{\mathcal{O}_X}(\mathcal{O}_X, \mathcal{V})) \\ &= \operatorname{Hom}_{\mathcal{O}_x}(\mathcal{O}_X \otimes_{R(X)} M, \mathcal{V}) \\ &= \operatorname{Hom}_{\mathcal{O}_X}(\Delta(M), \mathcal{V}) \end{aligned}$$

This shows that Δ is the left adjoint of Γ, so we have the adjointness morphism $M \longrightarrow \Gamma(X, \Delta(M))$ which induces the red arrows in the diagram above. By the five lemma, the adjointness morphism $M \longrightarrow \Gamma(X, \Delta(M))$ is an isomorphism.

Now consider the other adjointness morphism $\Delta(\Gamma(X, \mathcal{V})) \longrightarrow \mathcal{V}$. We have the exact sequence

$$0 \longrightarrow \mathcal{K} \longrightarrow \Delta(\Gamma(X, \mathcal{V})) \longrightarrow \mathcal{V} \longrightarrow \mathcal{C} \longrightarrow 0$$

where $\mathcal{K}$ is the kernal of the adjointness morphism and $\mathcal{C}$ is the cokernal. Since Γ is exact by Lemma 2.3, we can apply Γ to our sequence and exactness is preserved.

$$0 \longrightarrow \Gamma(X, \mathcal{K}) \longrightarrow \Gamma(X, \Delta(\Gamma(X, \mathcal{V})) \longrightarrow \Gamma(X, \mathcal{V}) \longrightarrow \Gamma(X, \mathcal{C}) \longrightarrow 0$$

Since $\Gamma \circ \Delta = \mathrm{id}$, the middle arrow in the sequence above is an isomorphism. This implies that

$$\Gamma(X, \mathcal{K}) = \Gamma(X, \mathcal{C}) = 0.$$

By Lemma 2.4, $\mathcal{K}$ and $\mathcal{C}$ are generated by their global sections, so $\mathcal{K} = \mathcal{C} = 0$. Thus, the second adjointness morphism is an isomorphism as well, and the functors Γ and Δ are mutually quasiinverse. □

Serre's theorm is powerful, but it only holds for affine varieties.

Example 2.6. Let $X = \mathbb{P}^n$. In this case, $\Gamma(X, \mathcal{O}_X) = \mathbb{C}$, since global sections of X are holomorphic functions on a compact complex manifold, so they cannot have local extrema. This implies that $\mathcal{M}(R(X))$ is isomorphic to the category of vector spaces. However, the other category $\mathcal{M}_{qc}(\mathcal{O}_X)$ is much richer than just the category of vector spaces, so we cannot possibly have an equivalence of categories.

As we can see from our example, Serre's Theorem does not hold for projective varieties. But if we expand our view to D-modules, we can generalize Serre's theorem to certain projective varieties.

3. Differential Operators on Affine Varieties

As before, let X be an affine variety over $\mathbb{C}$. We have a natural inclusion

$$R(X) \hookrightarrow \mathrm{End}_{\mathbb{C}}(R(X))$$

$$f \mapsto (g \mapsto fg)$$

where an element $f \in R(X)$ is associated with the left multiplication by f.

Definition 3.1. We say the $T \in End_{\mathbb{C}}(R(X))$ is a *differential operator* on X of *order* $\leq p$ if for any $f_0, f_1, \dots, f_p \in R(X)$,

$$[\dots[[T, f_0], \dots, f_p] = 0$$

Since $R(X)$ is commutative, $[f, g] = 0$ for all $f, g \in R(X)$, so $f \in R(X)$ is a differential operator of order ≤ 0.

Definition 3.2. We say that $T \in \mathrm{End}_{\mathbb{C}}(R(X))$ is a *derivation* (or *vector field*) on X if

$$T(fg) = T(f)g + fT(g).$$

If T is a derivation, then

$$[T, f](g) = T(fg) - fT(g) = T(f)g + fT(g) - fT(g) = T(f)g$$

so $[T, f] = T(f) \in R(X)$ and T is a differential operator of order ≤ 1. Let F_pD_X be the space of all differential operators on X of order $\leq p$. This is a linear space over $\mathbb{C}$.

Definition 3.3. We call

$$D_x = \bigcup_{p \in \mathbb{Z}} F_pD_X$$

the *space of differential operators* on X. This is a linear space over $\mathbb{C}$ containing $R(X)$.

Notice that if T and S are differential operators of order $\leq p$ and $\leq q$, respectively, then $T \circ S$ is a differential operator of order $\leq p + q$. This shows that D_X is a filtered ring.

4. Serre's Theorem for D-modules

We can sheafify this construction and get $\mathcal{D}_X$, the sheaf of differential operators on X. This is a sheaf of filtered rings. We have that $\Gamma(X, \mathcal{D}_X) = D_X$, and $\mathcal{O}_X \subset \mathcal{D}_X$. Let $\mathcal{M}(D_X)$ be the category of D_X-modules and $\mathcal{M}(\mathcal{D}_X)$ be the category of $\mathcal{D}_X$-modules. There are natural forgetful functors:

$$\mathcal{M}(D_X) \longrightarrow \mathcal{M}(R(X))$$

$$\mathcal{M}(\mathcal{D}_X) \longrightarrow \mathcal{M}(\mathcal{O}_X)$$

Let $\mathcal{M}_{qc}(\mathcal{D}_X) \subset \mathcal{M}(\mathcal{D}_X)$ be the full subcategory of sheaves of $\mathcal{D}_X$-modules which are quasicoherent as $\mathcal{O}_X$-modules. In this setting we still have natural global sections and localization functors

$$\Gamma(X, -)\colon \mathcal{M}_{qc}(\mathcal{D}_X) \longrightarrow \mathcal{M}(D_X)$$

$$\Delta\colon \mathcal{M}(D_X) \longrightarrow \mathcal{M}_{qc}(\mathcal{D}_X)$$

Where in this context, $\Delta(V) = \mathcal{D}_X \otimes_{D_X} V$. Remark 2.5 in the proof of Serre's theorem and the calculation after it hold for $\mathcal{D}_X$ and D_X-modules by identical arguments, so Γ and Δ are still adjoint functors when applied to $\mathcal{M}(D_X)$ and $\mathcal{M}_{qc}(\mathcal{D}_X)$. We can also note that Serre's two lemmas hold in this setting as well. Indeed, since $\Gamma\colon \mathcal{M}_{qc}(\mathcal{O}_X) \longrightarrow \mathcal{M}(R(X))$ is exact, and quasicoherent $\mathcal{D}_X$-modules are defined to be $\mathcal{D}_X$-modules which are quasicoherent as $\mathcal{O}_X$-modules, $\Gamma\colon \mathcal{M}_{qc}(\mathcal{D}_X) \longrightarrow \mathcal{M}(D_X)$ is exact on quasicoherent $\mathcal{D}_X$-modules. Furthermore, since a quasicoherent $\mathcal{D}_X$-module is generated by global sections as an $\mathcal{O}_X$-module, it must be generated by global sections as a $\mathcal{D}_X$-module as well. So the same proof that we used for Serre's theorem applies in this context, and we obtain the following result.

Theorem 4.1. *We have an equivalence of categories*

$$\mathcal{M}_{qc}(\mathcal{D}_X) \longleftrightarrow \mathcal{M}(D_X)$$

5. Leaving the Affine Case

The theorem at the end of the previous section doesn't solve our initial problem of Serre's Theorem only applying to affine varieties, since all of our definitions so far have been for affine X. However, we can extend all of our previous definitions to arbitrary varieties over $\mathbb{C}$ by glueing. When we do this, we have a ring D_X of differential operators for an arbitrary complex variety X.

Remark 5.1. In general, D_X is a "bad" ring - it is not noetherian and not generated by vector fields. However, we can restrict our assumptions on the variety X to obtain the properties we desire.

The following example shows the best possible case.

Example 5.2. Let $X = \mathbb{C}^n$. Define

$$I = (i_1, \ldots, i_n) \in \mathbb{Z}^n_{\geq 0}$$

$$z^I = z_1^{i_1} \cdots z_n^{i_n}$$

$$\partial^I = \partial_1^{i_1} \cdots \partial_n^{i_n}$$

where ∂_i is the partial derivative with respect to z_i. Then

$$D_X = \{T = \sum c_{IJ} z^I \partial^J | c_{IJ} \in \mathbb{C}\}.$$

In this case, D_X is left and right noetherian, and is generated over $R(X) = \mathbb{C}[z_1, \ldots, z_n]$ by $\partial_1, \ldots, \partial_n$.

These properties hold locally for any smooth variety X. In particular, we have the following theorem.

Theorem 5.3. *(Bernstein) Let X be a smooth variety, and $x \in X$. There exists an open affine neighborhood U containing x with*

i $f_1, \ldots, f_n \in R(U)$
ii vector fields $\partial_1, \ldots \partial_n$ *on* U *such that* $[\partial_i, \partial_j] = 0$
iii $\partial_i(f_j) = \delta_{ij}$

(This is sometimes referred to as a coordinate system on X.) Then

$$D_U = \{T = \sum f_I \partial^I | f_I \in R(U)\}$$

This theorem shows us that for the machinery of D-module theory to be useful, we need to consider smooth varieties. From this point on, we will assume that our variety X is smooth. In our next example, we revisit the case of $X = \mathbb{P}^n$ and see the extra structure that emerges when we expand our view from $\mathcal{O}_X$-modules to $\mathcal{D}_X$-modules.

Example 5.4. Let $X = \mathbb{P}^n$. As we discussed earlier, $\Gamma(X, \mathcal{O}_X) = \mathbb{C}$, so it is too small for our equivalence of categories to possibly hold. However, $\Gamma(X, \mathcal{D}_X)$ is much bigger. (In particular, it is infinite dimensional.) Surprisingly, an analogue of Serre's theorem holds in this case. We have an equivalence of categories

$$\mathcal{M}_{qc}(\mathcal{D}_X) \longleftrightarrow \mathcal{M}(D_X).$$

This equivalence generalizes Serre's Theorem for a specific projective variety. It allows us to "localize" D_X-modules.

More generally, we can extend Serre's Theorem for a whole class of projective varieties - flag varieties of complex semisimple Lie algebras. Let $\mathfrak{g}$ be a complex semisimple Lie algebra (e.g. $sl(n, \mathbb{C})$, the set of n by n traceless complex matrices). Let $\mathfrak{b}$ be a Borel subalgebra (e.g. upper triangular matrices in $sl(n, \mathbb{C})$). Let X be the flag variety of $\mathfrak{g}$ (e.g. the set of all complete flags in $\mathbb{C}^n$). Then we have an equivalence of categories:

$$\mathcal{M}_{qc}(\mathcal{D}_X) \longleftrightarrow \mathcal{M}(D_X)$$

This allows us to generalize Serre's Theorem for a certain class of projective varieties. It also leads to some important results in representation theory.

6. Connection to Representation Theory

In the case where X is the flag variety of a complex semisimple Lie algebra, we can describe D_X more explicitly in terms of representation theory. Let G be a connected complex algebraic group with Lie algebra $\mathfrak{g}$ (e.g. $G = SL(n, \mathbb{C})$). In this case, G acts on X algebraically; i.e. the action map

$$G \times X \longrightarrow X$$

$$(g, x) \mapsto g \cdot x$$

is a morphism of algebraic groups.

Definition 6.1. The *universal enveloping algebra* of $\mathfrak{g}$ is

$$\mathcal{U}(\mathfrak{g}) = T(\mathfrak{g}) \big/ (X \otimes Y - Y \otimes X - [X, Y])$$

where $T(\mathfrak{g})$ is the tensor algebra of $\mathfrak{g}$.

We can differentiate the action of G on X to get an action of $\mathfrak{g}$ on $S = \{\text{tangent vector fields on } X\}$. We can then extend this action to an action of the universal enveloping algebra on D_X. This gives us the following commutative diagram.

$$\begin{array}{ccc} \mathcal{U}(\mathfrak{g}) & \twoheadrightarrow & D_X \\ \big\uparrow & & \big\uparrow \\ \mathfrak{g} & \longrightarrow & S \end{array}$$

Let $\mathfrak{z}(\mathfrak{g})$ be the center of $\mathcal{U}(\mathfrak{g})$, and define $\mathcal{U}(\mathfrak{g})_0 = \mathfrak{g}\mathcal{U}(\mathfrak{g})$ (the kernal of the trivial representation). Then $I = \mathfrak{z}(\mathfrak{g}) \cap \mathcal{U}(\mathfrak{g})_0$ is a maximal ideal in $\mathfrak{z}(\mathfrak{g})$. Define $\mathcal{U}_0 = \mathcal{U}(\mathfrak{g}) \big/ I\mathcal{U}(\mathfrak{g})$. Then we have the following theorem.

Theorem 6.2. *$\mathcal{U}_0$ and D_X are isomorphic rings.*

From this, our result at the end of the previous section becomes the following theorem.

Theorem 6.3. *We have an equivalence of categories*

$$\mathcal{M}_{qc}(\mathcal{D}_X) \longleftrightarrow \mathcal{M}(\mathcal{U}_0)$$

This is a special case of a more general theorem of Beilinson and Benstein which is fundamental in geometric representation theory. $\mathcal{U}_0$-modules are special types of $\mathcal{U}(\mathfrak{g})$-modules, which tell us about the representation theory of G. So this theorem allows us to study representation theory problems using the "local" techniques of D-modules, which opens up a new world of geometric tools that can be used in the study of representation theory.