Aim: To investigate the hanging cable (catenary) problem (LN§ 1.1.1 on page 1) using the Shooting Method with trial and error zooming (LN§§ 2.1, 2.2 on page 3 ff).

1. An overhead power line is to be suspended between two points A and B at height 40 m above ground level, and 300 m apart horizontally. The cable has a line density $\rho = 0.8$ kg/m and is to be operated with a tension $\tau = 1450$ g N at its lowest point. Here $g = 9.8$ ms$^{-2}$ is the acceleration due to gravity. The aim of this exercise is to find the shape and various other characteristics of the cable.

If the cable is assumed perfectly flexible, then the height $h(x)$ of the cable as a function of horizontal distance $x$ is well-modelled by

$$\frac{d^2h}{dx^2} = \mu \sqrt{1 + \left(\frac{dh}{dx}\right)^2},$$

where $\mu = \rho g/\tau$. Although this problem superficially depends on three parameters, they occur in the dimensionless combination $\mu$, so it is only that particular combination that matters.

(a) Taking the coordinates of A to be $x = 0$, $h = 40$, draw a rough sketch of the cable configuration and write down 2 boundary conditions for $h(x)$. Equation (1) and these BC’s comprise a 2nd-order BVP for the height $h(x)$. To apply the Shooting Method, rewrite (1) as a 1st-order system of 2 d.e.’s in new variables $y_1 = h(x)$, $y_2 = h'(x)$, and write down initial conditions for $y_1$ and $y_2$, using an unknown $s$ to represent the slope $y_2$ at $x = 0$.

$$\frac{dy_1}{dx} = \ldots \quad \frac{dy_2}{dx} = \ldots$$

$$y_1(0) = \ldots \quad y_2(0) = \ldots$$

The d.e.’s (2) with IC’s (3) constitute a 1st-order IVP.

(b) The aim in this part is to find the shape of the cable. The BVP defined by (1) and the end-point conditions in (a) is similar to the example done in lectures, and can be solved by a MATLAB script similar to that in LN§2.2 on page 7. Files $bvp2/cable.inc$ and $bvp2/cable.f.inc$ contain incomplete programs. Copy them under the names cable.m and cable.f.m and edit to suit, i.e. complete where the ‘...’ appears. Then proceed as follows.

Run cable.m and enter guesses for $s_{\text{min}}$ and $s_{\text{max}}$ that you think might bound the slope $dh/dx$ at the LH end. (Hint: think of a real power line. What kinds of slopes do the cables have as they come off the pylons?) Try to straddle the given boundary value of $h$ at the RH end with the 6 shots taken by cable.m. Once you succeed, refine the $s$-range and rerun cable.m. Repeat the refinement, stopping when you cannot visually distinguish the different cable shapes on your graph. Then, from your output table estimate the correct left slope. [Ans: $h'(0) \approx -0.0829$, although this is probably only significant to the first leading non-zero digit, i.e. to the 2nd decimal place, given the small number of significant digits in $g$ and $\rho$.]
(c) Use your plot of $h$, and MATLAB’s zoom command, to estimate the height of the lowest point. [Just enter zoom and repeatedly left-click on the lowest point of your graph.] Hence calculate the sag of the cable. [Ans: $h(150) \simeq 33.78\text{m}$, sag is $40 - h(150)$.

(d) Find the total length $L$ of the cable. To do this, use the formula for the arc length of a curve $y(x)$ between $x = a$ and $x = b$, i.e.

$$
\int_a^b \sqrt{1 + \left(\frac{dy}{dx}\right)^2} \, dx.
$$

In cable.m, $dh/dx$ is just $yvec(:,2)$, the second column of the solution array $yvec$. This can be used as input to the MATLAB numerical integration routine trapz, which applies the trapezoidal integration rule to evenly spaced points, as shown below.

```matlab
integrand=sqrt(1 + ...);
L=trapz(xspan,integrand)
```

Thus find $L$. Note how little extra cable is used by the sag. [Ans: $L \simeq 300.343$; extra length about $0.3$ m for a sag of about $6$ m.]

(e)* Plot the tension $T$ throughout the cable against $x$. Note that $T \cos \theta = \tau$ is constant, where $\tan \theta = dh/dx$. Why? Find the maximum tension. [Ans: $T_{\text{max}} \simeq 1.426\times10^4\text{N.}$] It is attained at the end-points. Why?

## Additional Practice Exercises

2. Consider the BVP

$$
y'' + yy' = e^{x^2/2}, \quad y(-1) = 2, \quad y(1) = 3.
$$

(a) Rewrite this BVP as a 1st order BVP system.

(b) Write out a corresponding 1st order IVP, suitable for the Shooting Method, shooting from $x = -1$.

(c) Copy cable.m and cable f.m under names like bvptut2q2.m and bvptut2q2 f.m. Edit these to suit, and hence use MATLAB to determine a good estimate for the missing initial condition, and to plot the corresponding solution. [Ans: $y'(-1) \simeq 0.911$.]

3.* Repeat Q2 but for the BVP

$$
\frac{d^3y}{dt^3} + 10\frac{dy}{dt} - 5y^3 = t^3 - ty, \quad y(0) = 0, \quad y(2) = -1, \quad y''(2) = 0.
$$

Here, the missing initial condition is at the RH end, so the shooting should be done from right to left. [See LN§2.3.2 on page 11. Ans: $y'(2) \simeq -0.499$.]

4.* For hanging cables or chains of the type in Q1, the shape curve is known as a catenary (from the Latin word catena for chain). The problem of determining the catenary shape was proposed by Galileo (1564–1642). The mathematical form was stated without proof by Leibnitz (1691) and verified using (1) and differential calculus by James Bernoulli (1691). Referred to coordinates $(X,Y)$ with origin at the lowest point, the catenary equation is

$$
Y(X) = c\left(\cosh \frac{X}{c} - 1\right).
$$

Find the relations between $X$, $Y$ and $x$, $h$ in Q1, and find the relation between $c$ and $\mu$ such that (1) is satisfied. Modify cable.m so as to superimpose the exact solution $Y$ on the original plot, using ‘o’ to show $Y$ at 7 equispaced points (including the ends).