1. Find the **surface area** of the part of the sphere $x^2 + y^2 + z^2 = 25$ that lies between the planes $z = 3$ and $z = 4$.

**Solution:** Let $S$ be the part of the sphere between the planes $z = 3$ and $z = 4$. Let $R$ be the projection of $S$ onto the $xy$-plane. So $R$ is given by $9 \leq x^2 + y^2 \leq 16$
Surface area of $S = \int \int_S dS = \int \int_R \sqrt{f_x^2 + f_y^2 + 1} \, dx \, dy,$

where $z = f(x, y) = \sqrt{25 - x^2 - y^2}$.

So $f_x = -\frac{x}{\sqrt{25 - x^2 - y^2}}, \quad f_y = -\frac{y}{\sqrt{25 - x^2 - y^2}},$

and $\sqrt{f_x^2 + f_y^2 + 1} = \frac{5}{\sqrt{25 - x^2 - y^2}}$.

Hence, the surface area of $S = \int \int_R \frac{5}{\sqrt{25 - x^2 - y^2}} \, dx \, dy$.

Using polar coordinates, $R$ is described by $x = r \cos \theta, \ y = r \sin \theta,$

with $0 \leq \theta \leq 2\pi$ and $3 \leq r \leq 4$. Therefore, the surface area is

$$\int_0^{2\pi} \int_3^4 \frac{5}{\sqrt{25 - r^2}} \, r \, dr \, d\theta = \int_0^{2\pi} -5\sqrt{25 - r^2} \bigg|_3^4 \, d\theta$$

$$= \int_0^{2\pi} 5 \, d\theta = 10\pi.$$
2. Find the surface area of the part of the paraboloid \( z = 10 - x^2 - y^2 \) which lies above the \( xy \)-plane.

**Solution:** Let \( S \) be the surface of the paraboloid, and \( R \) the projection of \( S \) onto the \( xy \)-plane. So \( R \) is given by \( x^2 + y^2 \leq 10 \)

Surface area of \( S = \iint_S dS \)

\[
= \iint_R \sqrt{f_x^2 + f_y^2 + 1} \, dx \, dy,
\]

where \( z = f(x, y) = 10 - x^2 - y^2 \).

So \( f_x = -2x \); \( f_y = -2y \)

and \( \sqrt{f_x^2 + f_y^2 + 1} = \sqrt{4x^2 + 4y^2 + 1} \).

Hence, surface area = \( \iint_R \sqrt{4x^2 + 4y^2 + 1} \, dx \, dy \)
Using polar coordinates, \( R \) is described by \( x = r \cos \theta, \ y = r \sin \theta \), with \( 0 \leq \theta \leq 2\pi \) and \( 0 \leq r \leq \sqrt{10} \). Therefore, the

\[
\text{surface area} = \int_0^{2\pi} \int_0^\sqrt{10} \sqrt{4r^2 + 1} r \ dr \ d\theta \quad \text{(via polars)}
\]

\[
= 2\pi \times \frac{1}{8} \times \frac{2}{3} (4r^2 + 1)^{3/2} \bigg|_0^\sqrt{10} = \frac{(41^{3/2} - 1)\pi}{6}.
\]

3. Let \( S \) be the triangular portion of the plane \( 3x + 3y + 5z = 30 \) in the first octant — that is, the portion of the plane cut off by the planes \( x = 0, \ y = 0 \) and \( z = 0 \), for \( x \geq 0, \ y \geq 0 \) and \( z \geq 0 \).

(i) Sketch the region \( S \).

(ii) Let \( R \) be the projection of \( S \) onto the \( xy \)-plane. Describe \( R \), in terms of \( x \) and \( y \).

(iii) Suppose that a thin plate in the shape of \( S \) has density \( (x + y + z) \) at each point \( (x, y, z) \). Find the mass of the plate.

\[
(\text{Mass} = \iint_S (x + y + z) \ dS.)
\]

Solution:
(i) The region $S$ is as sketched below:

(ii) $R$ is described by $0 \leq x \leq 10; 0 \leq y \leq 10 - x.$ and is as sketched above.
(iii) Recall that

\[ \iint_S \phi(x, y, z) \, dS = \iint_R \phi(x, y, f(x, y)) \sqrt{f_x^2 + f_y^2 + 1} \, dx \, dy \]

where \( S \) is a surface given by the equation \( z = f(x, y) \) and \( R \) is the projection of \( S \) onto the \( xy \)-plane.

Mass of \( S \) = \[ \iint_S (x + y + z) \, dS \]

= \[ \iint_R (x + y + z) \sqrt{f_x^2 + f_y^2 + 1} \, dx \, dy, \]

where \( z = f(x, y) = \frac{30 - 3x - 3y}{5} \).

So \[ \sqrt{f_x^2 + f_y^2 + 1} = \sqrt{\left(\frac{-3}{5}\right)^2 + \left(\frac{-3}{5}\right)^2 + 1} = \frac{\sqrt{43}}{5}. \]
Mass of $S = \int\int_{S} (x + y + z) \, dS$

\[= \int\int_{R} \left( x + y + \frac{30 - 3x - 3y}{5} \right) \frac{\sqrt{43}}{5} \, dx \, dy \]

\[= \frac{\sqrt{43}}{25} \int_{0}^{10} \int_{0}^{10-x} (2x + 2y + 30) \, dy \, dx \]

\[= \frac{\sqrt{43}}{25} \int_{0}^{10} \left[ 2xy + y^2 + 30y \right]_{0}^{10-x} \, dx \]

\[= \frac{\sqrt{43}}{25} \int_{0}^{10} (2x(10 - x) + (10 - x)^2 + 30(10 - x)) \, dx \]

\[= \frac{\sqrt{43}}{25} \times \frac{6500}{3} = \frac{1300\sqrt{43}}{15}. \]

4. Let $S$ be the surface defined by the following set of points in $\mathbb{R}^3$:

\[\{(x, y, z) \mid x + z = 5, \ 0 \leq x \leq 3, \ 0 \leq y \leq 4\}. \]

(i) Sketch $S$, and its projection onto the $xy$-plane.

(ii) Find the cost of painting this surface if the cost is $$(xy + z^2)$$
per unit area. (Cost = \( \int \int_{S} (xy + z^2) \, dS \).)

**Solution:**

(i) \( S \) is the plane shown. Let \( R \) be the projection of \( S \) onto the \( xy \)-plane. \( R \) is the rectangular region \( 0 \leq x \leq 3, \ 0 \leq y \leq 4 \).
(ii) The equation of $S$ can be rewritten as $z = f(x, y) = 5 - x$.

So $f_x = -1$ $f_y = 0$ and $\sqrt{f_x^2 + f_y^2 + 1} = \sqrt{2}$.

On $S$: $xy + z^2 = xy + (5 - x)^2$.

Cost of painting $= \int\int_S (xy + z^2) \, dS$

$= \int\int_R (xy + (5 - x)^2) \sqrt{f_x^2 + f_y^2 + 1} \, dx \, dy$

$= \int_0^4 \int_0^3 (xy + (5 - x)^2) \sqrt{2} \, dx \, dy$

$= \sqrt{2} \int_0^4 \left[ \frac{x^2y}{2} - \frac{(5 - x)^3}{3} \right]_0^3 \, dy$

$= \sqrt{2} \int_0^4 \left( 39 + \frac{9y}{2} \right) \, dy = 192\sqrt{2}$. 