1. For each of the following sequences of vertices, state whether or not it represents a walk, trail, path, closed walk, closed trail, or cycle in the graph illustrated.

(i) $abdefcbd$
(ii) $abdefcd$
(iii) $abdefcdab$
(iv) $beafcd$
(v) $bcdb$
(vi) $abefcd$

**Solution.**

A **walk** is a finite sequence of edges of the form $v_1v_2, v_2v_3, v_3v_4, \ldots, v_{n-1}v_n$ in which any two consecutive edges are adjacent or identical. If $v_1 = v_n$ the walk is **closed**.

A **trail** is a walk in which all the edges are distinct.

A **path** is a trail in which all the vertices (except possibly $v_1$ and $v_n$) are distinct.

A closed path (with at least one edge) is a **cycle**.

(i) A walk. (ii) A trail (and also a walk). (iii) A closed walk.
(iv) A closed trail (and also a closed walk).
(v) A cycle (and a closed trail, and a closed walk).
(vi) This is not a walk of any type.

2. Suppose that a graph $G$ is regular of degree $r$, where $r$ is odd.

(i) Prove that $G$ has an even number of vertices.

(ii) Prove that the number of edges in $G$ is a multiple of $r$.

**Solution.**

(i) Let $n$ be the number of vertices. Then, since each vertex has degree $r$, the sum of the degrees is $n \times r$, and this must be an even number (by the hand-shaking lemma). Now, $r$ is odd, and so $n$ must be even.

(ii) The number of edges is $\frac{n \times r}{2}$. (The hand-shaking lemma again.)

Since $n$ is even, $n = 2k$ for some integer $k$, and hence the number of edges is $k \times r$.

3. Let $u$ and $v$ be distinct vertices of a graph. Prove that there is a walk from $u$ to $v$ if and only if there is a path from $u$ to $v$.

**Solution.**

Since a path is a walk (without repeated vertices), if there is a path from $u$ to $v$ then there is a walk from $u$ to $v$. 
Now suppose that there is a walk \( W \) from \( u \) to \( v \) which is not a path – that is, a walk containing repeated vertices. Between any two occurrences of a repeated vertex in \( W \), there is a closed walk. Remove from \( W \) all such closed walks, and the remaining edges will form a path from \( u \) to \( v \).

4. A simple graph has 20 vertices. Any two distinct vertices \( u \) and \( v \) are such that \( \text{deg}(u) + \text{deg}(v) \geq 19 \). Prove that the graph is connected.

Solution.

Suppose the graph is disconnected, with \( k \geq 1 \) vertices in one component, and \( 20 - k \) vertices in another. Then the maximum possible degree for a vertex in the first component is \( k - 1 \), and in the second is \( 19 - k \). Therefore, if a vertex \( u \) is in the first component, and a vertex \( v \) is in the other component, \( \text{deg}(u) + \text{deg}(v) \leq (k - 1) + (19 - k) = 18 \). But \( \text{deg}(u) + \text{deg}(v) \geq 19 \) for all distinct vertices \( u \) and \( v \). We have a contradiction, and so the graph must be connected.

5. (i) Draw the graphs formed by the vertices and edges of a tetrahedron, a cube, and an octahedron.

(ii) Are any of these graphs Eulerian?

(iii) Find a Hamiltonian cycle in each graph.

Solution.

(i) Tetrahedron: Cube: Octahedron:

(ii) A connected graph is Eulerian if there is a closed trail which uses every edge exactly once.

A connected graph is semi-Eulerian if it is not Eulerian, but there is a trail which uses every edge exactly once.

A connected graph is Eulerian if and only if the degree of each vertex is even.

So the graph of the octahedron is the only one which is Eulerian.

(iii) A graph is Hamiltonian if there exists a closed path which passes through every vertex of the graph exactly once. Such a path is called a Hamiltonian cycle.
The solid lines in the graphs below are Hamiltonian cycles.

6. (a) Is it possible to draw a sketch of \(K_5\) without lifting your pen from the paper, and without retracing any edges?
(b) For which values of \(n \geq 2\) is \(K_n\) (i) Eulerian? (ii) semi-Eulerian?

**Solution.**
(a) The degree of each vertex in \(K_5\) is 4, and so \(K_5\) is Eulerian. Therefore it can be sketched without lifting your pen from the paper, and without retracing any edges.

(b) (i) In \(K_n\) the degree of each vertex is \(n - 1\). A graph is Eulerian if and only if the degree of each vertex is even. Therefore, \(K_n\) is Eulerian if \(n\) is odd.

(ii) The only semi-Eulerian complete graph is \(K_2\).

7. Find, if possible, an Euler trail or a semi-Euler trail in this graph:

**Solution.**
The graph is connected, and there are exactly two vertices of odd degree. So there is no Euler trail, but there is a semi-Euler trail beginning at one of the odd vertices, and ending at the other.

Add an edge (\(jk\) in the diagram below) between the two vertices of odd degree to make the graph Eulerian, and apply the algorithm using the removal of a cycle to find an Euler trail.

Find a closed trail \(T\), and remove from the graph the edges of \(T\), and any vertices which become isolated as a result. For example, in the following diagram, the trail \(T = acfkligfeca\) has been chosen. Vertices \(a, f\) and \(l\) become isolated, and are removed.

Call the subgraph which remains \(H\). Now find an Euler trail in each of the components of \(H\). In this case, it’s easy enough to see the following Euler trails:
In the left-hand component: $cdebc$

In the right-hand component: $ghjkihg$

Now choose any vertex of $T$, and follow $T$ until we reach a vertex of $H$. Suppose we start at $c$. Since $c$ is also a vertex of $H$, follow the entire Euler trail in the left-hand component of $H$ back to $c$, then follow $T$ until another vertex of $H$ is reached. In this case, the vertex is $k$. Follow the Euler trail in the right-hand component of $H$ back to $k$, and then follow $T$ back to $c$. We then have the Euler trail

$cdebcfkikhkgjkligfeac$.

Removing the edge $fk$ we obtain a semi-Euler trail, starting at $k$ and finishing at $f$:

$kikhghjkligfeacdebcf$.

**Alternate solution**

Using Fleury’s algorithm, we must start at $f$ or $k$ ($f$, say), and make a trail, noting at each step which edge is used, and regarding it as being removed from the graph. **However, we must never use an edge, use of which would leave us disconnected from any of the remaining unused edges.** Thus, at step 1, we must not use the edge $[f, g]$. Either $[f, c]$ or $[f, e]$ will do. (This choice means the trail will not be unique.)

Let us choose, say, $[f, c]$. We are then at vertex $c$, and, having removed edge $[f, c]$, we have (second diagram):

Now we can choose $[c, a]$ or $[c, b]$ or $[c, d]$, none of which disconnect us from any unused (undotted) edges. Choose $[c, a]$, say, in which case we must follow with $[a, e]$. We are now at $e$ (last diagram):

Continuing this way (avoiding such mistakes as now using $[e, f]$, which would leave us disconnected from some unused edges), we will eventually use up all the edges, ending at $k$. (Note, however, that we will have visited $k$ twice beforehand, and that we will have revisited $f$ once. This is no problem, as a trail being Euler has nothing to do with how many times any vertices are used.)

One semi-Euler trail is that which visits the vertices in the following order:

$fcaebcdefghikjhgilk$. 
8. (a) A domino is a rectangular tile containing a number of dots (i say) near one end, and a number of dots (j say) near the other end. Call this an [i, j] type domino. Is it possible to arrange 10 dominoes, types [1, 2], [1, 3], [1, 4], [1, 5], [2, 3], [2, 4], [2, 5], [3, 4], [3, 5], [4, 5], end to end so that the numbers of dots on touching ends of adjacent dominoes are always equal?

(b) Repeat the above question in the more general case, when there are \( \binom{n}{2} \) dominoes \( (n \geq 2) \), types all combinations \([i, j]\) with \( 1 \leq i < j \leq n \).

Solution.

(a) Let \( G \) be the graph whose vertices are 1, 2, 3, 4, 5, and whose edges are the domino types (i.e., \( i \) is adjacent to \( j \) when there is a domino of type \([i, j]\) or \([j, i]\)).

Clearly \( G = K_5 \), the complete graph on 5 vertices (all pairs of vertices adjacent). The question is whether or not an Euler trail (or a semi-Euler trail) exists in \( G \). Since all vertices in \( G \) have even degree, there is an Euler trail, so the answer is yes.

For example, the trail which takes the vertices in the order

\[ 1, 2, 3, 4, 5, 1, 3, 5, 2, 4, 1 \]

is an Euler trail, which uses the edges in the order

\[ [1, 2], [2, 3], [3, 4], [4, 5], [5, 1], [1, 3], [3, 5], [5, 2], [2, 4], [4, 1] \]

and which gives an arrangement of the dominoes satisfying the required condition.

(b) Similarly as in part (a), we form \( K_n \), the complete graph on \( n \) vertices, which is regular of degree \( n - 1 \) (all vertices having degree \( n - 1 \)). Hence a solution certainly exists if \( n - 1 \) is even, i.e., if \( n \) is odd, since then all vertices have even degree and an Euler trail exists, giving an arrangement of the dominoes in the required way.

If \( n \) is 2, there is only one domino, and the result is trivially true — the arrangement is a row containing just that one domino. This corresponds to the fact that \( K_2 \) has 2 vertices of odd degree, so a semi-Euler trail exists.

If \( n \) is even but greater than 2, \( K_n \) has more than 2 vertices of odd degree, so no semi-Euler trail exists, and the dominoes cannot be arranged in the required manner.

(Note that this question is equivalent to question 6.)