1. In a tournament, the score of a vertex is its out-degree, and the score sequence is a list of all the scores in non-decreasing order.

(i) If the score sequence of a tournament is \((s_1, s_2, \ldots, s_n)\), show that \(\sum_{i=1}^{n} s_i = \frac{n(n - 1)}{2}\).

(ii) What is the score sequence of a tournament with \(n\) vertices if each vertex has a different score?

(iii) Find a semi-Hamiltonian path in a tournament with \(n\) vertices \(v_0, v_1, \ldots, v_{n-1}\) if the score of vertex \(v_i\) is \(i\).

**Solution.**

(i) In any digraph the sum of the out-degrees is equal to the sum of the in-degrees. (Every edge has to go out of one vertex and into another.) The sum of all the out-degrees and all the in-degrees is, of course, twice the number of edges (by the hand-shaking lemma). Therefore, the sum of the out-degrees is equal to the number of edges. Now, if the score sequence of a tournament is \((s_1, s_2, \ldots, s_n)\), then the tournament has \(n\) vertices, and

\[\sum_{i=1}^{n} s_i = \text{the sum of the out-degrees} = \text{the number of edges}.\]

But a tournament with \(n\) vertices has \(\frac{n(n - 1)}{2}\) edges, since each vertex is joined to every other vertex by exactly one directed edge. The result therefore follows.

(ii) The score, or out-degree, of any vertex in a tournament with \(n\) vertices is at most \(n - 1\). Clearly, the score of a vertex is not negative, so any score \(s\) is such that \(0 \leq s \leq n - 1\). The only set of \(n\) different integers in this range is \(\{0, 1, 2, \ldots, n - 1\}\). So the score sequence is \((0, 1, 2, \ldots, n - 1)\).

(iii) Consider vertex \(v_{n-1}\). Its out-degree is \(n - 1\), and so there is an edge directed from \(v_{n-1}\) to every other vertex, and in particular to \(v_{n-2}\). Vertex \(v_{n-2}\) has out-degree \(n - 2\), and in-degree \(1\) (from \(v_{n-1}\)). There is therefore an edge directed from \(v_{n-2}\) to each of \(v_0, v_1, \ldots, v_{n-3}\). In particular, there is an edge directed from \(v_{n-2}\) to \(v_{n-3}\). Continuing in this way, we see that for each \(i, 1 \leq i \leq n - 1\), there is an edge directed from \(v_i\) to \(v_{i-1}\). Therefore \(v_{n-1}v_{n-2}\ldots v_0\) is a path containing all the vertices – that is, a semi-Hamiltonian path.

2. The following are adjacency matrices for two digraphs. In each case, determine whether or not the digraph is acyclic by attempting to order the vertices in such a way that the adjacency matrix is upper triangular.
If the graph is acyclic, write the matrix as an upper triangular matrix. Otherwise, find a cycle in the graph.

(i) \[
\begin{array}{cccccccc}
A & B & C & D & E & F & G & H \\
A & 0 & 1 & 1 & 0 & 0 & 0 & 1 \\
B & 0 & 0 & 1 & 1 & 0 & 0 & 0 \\
C & 0 & 0 & 0 & 1 & 0 & 0 & 1 \\
D & 0 & 0 & 0 & 0 & 1 & 1 & 0 \\
E & 1 & 1 & 0 & 0 & 1 & 0 & 0 \\
F & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
G & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
H & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\
\end{array}
\]

(ii) \[
\begin{array}{cccccccc}
A & B & C & D & E & F & G & H \\
A & 0 & 1 & 1 & 0 & 1 & 0 \\
B & 0 & 0 & 0 & 0 & 1 & 1 \\
C & 0 & 1 & 0 & 0 & 0 & 1 \\
D & 1 & 1 & 1 & 0 & 1 & 1 \\
E & 0 & 0 & 0 & 0 & 0 & 0 \\
F & 1 & 0 & 0 & 0 & 1 & 0 \\
G & 0 & 0 & 0 & 0 & 0 & 1 \\
H & 0 & 0 & 0 & 0 & 0 & 0 \\
\end{array}
\]

Solution.

Vertices are ordered by finding a vertex of zero in-degree, calling it vertex 1, and then removing that vertex and all its edges. Then look for another vertex of zero in-degree, call it vertex 2, remove it and all its edges. Repeat the process until all the vertices are numbered. If at any stage there is not a vertex with zero in-degree then the digraph must have a cycle. The ordering can be done directly from the adjacency matrix, since a column of zeros corresponds to a vertex having zero in-degree.

(i) Vertex \(E\) has zero in-degree. After removing the row and column corresponding to vertex \(E\), we see that vertex \(A\) has zero in-degree. Remove the row and column corresponding to vertex \(A\), etc. The ordering which results is: \(E, A, B, C, D, H, G, F\), and the corresponding upper triangular matrix is:

\[
\begin{array}{cccccccc}
E & A & B & C & D & H & G & F \\
E & 0 & 1 & 1 & 0 & 0 & 0 & 1 \\
A & 0 & 0 & 1 & 1 & 0 & 0 & 1 \\
B & 0 & 0 & 0 & 1 & 1 & 1 & 0 \\
C & 0 & 0 & 0 & 0 & 1 & 1 & 1 \\
D & 0 & 0 & 0 & 0 & 1 & 1 & 1 \\
H & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
G & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\
F & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\end{array}
\]

(ii) Here we see that vertex \(D\) has zero in-degree. Removing vertex \(D\) and its edges leaves a graph in which no vertex has zero in-degree. The graph must therefore have a cycle. A cycle can be found simply by constructing the digraph and looking for one. There are, for example, two directed triangles: \(A \to B \to F \to A\) and \(A \to C \to F \to A\).

3. (i) Show that if there is a directed walk in a digraph between a pair of vertices \(u\) and \(v\), then there is a directed path between \(u\) and \(v\).

(ii) Show that any directed walk in an acyclic digraph is a directed path.
Solution.

(i) A path is a walk without repeated vertices.

(ii) Suppose a walk in an acyclic graph contains repeated vertices; for example, \( v_1 \rightarrow \cdots \rightarrow v_i \rightarrow \cdots \rightarrow v_i \rightarrow \cdots \rightarrow v_n \).

But then there is a cycle from \( v_i \) to \( v_i \), so such a walk cannot exist.

4. There are 15 computer programs that must be processed according to the following set of orders: 1 > 2, 7, 13; 2 > 3, 8, 14; 3 > 9, 15; 4 > 3; 5 > 4, 11; 6 > 5, 12; 7 > 6; 8 > 7, 9, 14; 9 > 15; 10 > 4, 9; 11 > 10; 12 > 11; 13 > 7, 12; 14 > 13, 15; where 1 > 2, 7, 13 means that programs 2, 7 and 13 can be processed only after program 1 has been processed.

Is it possible for the programs to be processed? If so, give a processing sequence.

Solution.

Consider the digraph with 15 vertices corresponding to the programs, and directed edges corresponding to the set of orders. (For example, 1 \(\rightarrow\) 2, 1 \(\rightarrow\) 7 and 1 \(\rightarrow\) 13 are edges.) Then the programs can be processed if the digraph is acyclic, and we can determine whether or not this is so by attempting to order the vertices as in question 2.

The ordering is possible, and a processing sequence is:

\[1, 2, 8, 14, 13, 7, 6, 5, 12, 11, 10, 4, 3, 9, 15.\]

5. Explain why the flow in the following network is not valid.

Solution.

On each arc the first number represents the capacity, and the second number the flow. For a valid flow, flow must be less than or equal to the capacity for each arc, and at each vertex other than the source and the sink, the total inflow must equal the total outflow. (The source is the vertex \( A \), with zero in-degree and the sink is the vertex \( Z \) with zero out-degree.)

In this network, on arc \( DE \) the flow is greater than the capacity. So the flow is not valid. Furthermore, at vertices \( B, C, D \) and \( E \) the inflow does not equal the outflow. (For example, at \( B \) the inflow is 4 and the outflow is 5.

6. (i) Verify that the flow in the following network is valid.
(ii) What is the value of the flow? (iii) Which arcs are saturated?
(iv) Find the capacity of each of the following cuts: \( \{A,B,C\}\mid\{D,E,Z\}\); \( \{A,C\}\mid\{B,D,E,Z\}\); \( \{A,C,E\}\mid\{B,D,Z\}\); \( \{A,E\}\mid\{B,C,D,Z\}\).

Verify that the flow across each of these cuts is equal to the value of the flow, and that in each case the capacity of the cut is greater than the flow across the cut.

(v) Find a flow-increasing path and hence find the maximum flow.

(vi) Find two cuts, each with capacity equal to the maximum flow.

**Solution.**

(i) On each arc, flow is less than or equal to capacity, and at each vertex other than the source and sink the inflow equals the outflow. Therefore the flow is valid.

(ii) The value of the flow is 5 (= outflow at A= inflow at Z.)

(iii) The saturated arcs are those for which flow equals capacity, that is, \(AB\), \(AC\), \(DZ\) and \(DE\).

(iv) A cut is a partitioning of the vertices into two subsets \(S\) and \(T\), so that \(S\) contains the source (and maybe some other vertices) and \(T\) contains the sink (and maybe some other vertices).

The capacity of a cut \(\{S\mid T\}\) is the sum of the capacities of arcs directed from \(S\) to \(T\).

For a given flow, the flow across a cut \(\{S\mid T\}\) is the sum of the flows in arcs directed from vertices in \(S\) to vertices in \(T\), minus the sum of the flows in arcs directed from vertices in \(T\) to vertices in \(S\).

The cut \(\{A,B,C\}\mid\{D,E,Z\}\):

![Diagram](https://example.com/diagram1.png)

Capacity of cut = 5 + 2 + 2 = 9, flow across the cut = 4 + 1 = 5.

The cut \(\{A,C\}\mid\{B,D,E,Z\}\):

![Diagram](https://example.com/diagram2.png)

Capacity of cut = 4 + 2 + 2 = 8, flow across the cut = 4 + 1 = 5.
The cut \( \{A, C, E\} \setminus \{B, D, Z\} \):

[Diagram of network flow with capacities shown]

Capacity of cut = \( 4 + 2 + 5 = 11 \), flow across the cut = \( 4 + 3 - 2 = 5 \).

The cut \( \{A, E\} \setminus \{B, C, D, Z\} \):

[Diagram of network flow with capacities shown]

Capacity of cut = \( 4 + 2 + 1 + 5 = 12 \),
flow across the cut = \( 4 + 1 - 1 - 2 + 3 = 5 \).

Note that in each case the flow across the cut is equal to the value of the flow, and that the capacity of each cut is greater than 5.

\((v)\)

Note that the arcs \( AB \) and \( AC \) are saturated, so the flow cannot be increased in those arcs. The flow in \( AD \) can be increased by 2, but these 2 cannot be taken along either \( DZ \) or \( DE \). However, 1 unit can be diverted from \( BD \) and taken down \( BC \), and thence along \( CE \) and \( EZ \). That is, there is a flow-increasing path \( A \rightarrow D \leftarrow B \rightarrow C \rightarrow E \rightarrow Z \), along which the flow can be increased by 1. Note that \( DB \) is a “backwards” flowing arc, and so the flow in \( BD \) is decreased by 1. We therefore have the following:

[Diagram of network flow with capacities shown]

No more flow-increasing paths can be found, and so we now have the maximum flow, with value 6.

\((vi)\) The cut \( \{A, B, C, D\} \setminus \{E, Z\} \) has capacity 6, as does \( \{A, B, D\} \setminus \{C, E, Z\} \). Note that both these cuts are across saturated arcs.