1. [15 marks]

(a) Suppose that the process \( \{X_t; \ t \in \mathbb{Z}\} \) is generated by \( X_t = Z_t - \theta_1 Z_{t-1} - \theta_2 Z_{t-2} \), where \( \{Z_0, Z_1, Z_2, \cdots\} \) is a sequence of iid random variables on \( Z \). Given that the moment generating function (mgf) of \( Z \), \( M(\lambda) = E[exp(\lambda Z)] \), find the constants \( \beta_k; \ k = 1, 2, \cdots, n + 2 \) in terms of \( \delta_1, \delta_2, \cdots, \delta_n, \theta_1 \) and \( \theta_2 \) such that \( \sum_{i=1}^{n} \delta_i X_{i+h} = \sum_{k=1}^{n+2} \beta_k Z_{h+k-2}, \ h \geq 0 \). Hence express the joint mgf \( E[exp(\sum_{i=1}^{n} \delta_i X_{i+h})] \) in terms of \( M(\cdot) \). Deduce that this joint mgf is independent of \( h \).

Comment on the ‘strict stationarity’ of the process \( \{X_t\} \).

(b) Suppose that \( Y_t \) and \( X_t \) are related by \( Y_t = \frac{1}{l} (X_t + X_{t-1} + \cdots + X_{t-l+1}) \), where \( \{Y_t\} \) and \( \{X_t\} \) two stationary processes with spectrums \( h_Y(\omega) \) and \( h_X(\omega) \) respectively. Write down the transfer function \( \alpha(\omega) \) for suitably chosen constants \( \alpha_j \) such that
\[
\alpha(\omega) = \sum_{j=0}^{l-1} \alpha_j e^{-i\omega j}.
\]
Show that
\[
|\alpha(\omega)|^2 = \frac{1 - \cos(l\omega)}{l^2[1 - \cos(\omega)]^2}.
\]
If the series \( \{X_t\} \) is repeatedly smoothed \( n \) times by taking a Moving Average of span 2, examine the behaviour of the spectrum of the series \( \{Y_t\} \) near \( \omega = 0 \) as \( n \to \infty \). Deduce that this approach may effectively remove all frequency components except at \( \omega = 0 \).

\ldots/2
2. [20 marks]

(a) A stationary process \( \{X_t\} \) is said to be an ARCH(1) if it satisfies

\[
X_t = \sigma_t \epsilon_t, \quad \sigma_t^2 = \alpha_0 + \alpha_1 X_{t-1}^2,
\]

where \( \{\epsilon_t\} \) is a sequence of iid random variables with mean zero and variance 1, \( \alpha_0 > 0 \) and \( \alpha_1 \geq 0 \).

i. Find \( E(X_t) \) and show that \( E(X_t^2) = \frac{\alpha_0}{1-\alpha_1} \). Hence give an upper bound for \( \alpha_1 \).

ii. Assuming \( X_t \) is fourth-order stationary, find an expression for \( E(X_t^4) \). Hence show that \( \alpha_1 \) must be in the region \( 0 \leq \alpha_1 < \frac{1}{a} \), where \( a^2 = E(\epsilon_t^4) \).

iii. Under the normality assumption, show that the unconditional kurtosis, \( \kappa = \frac{E(X_t^4)}{E(X_t^2)^2} \) is given by \( \frac{\alpha(1-\alpha)^2}{1-3\alpha_1} \). Deduce that the tail distribution of \( X_t \) is heavier than that of a normal distribution.

(b) Consider the GARCH \((p,q)\) process given by

\[
X_t = \sigma_t \epsilon_t, \quad \sigma_t^2 = \alpha_0 + \sum_{i=1}^{p} \alpha_i X_{t-i}^2 + \sum_{j=1}^{q} \beta_j \sigma_{t-j}^2,
\]

where \( \{\epsilon_t\} \) is defined as in (a) and \( \alpha_0 > 0, \alpha_i \geq 0, \beta_j \geq 0 \).

i. Let \( \eta_t = X_t^2 - \sigma_t^2 \). Show that \( \eta_t \) is a martingale difference series and \( X_t^2 \) satisfies an ARMA \((r,q)\) process such that

\[
X_t^2 = \alpha_0 + \sum_{i=1}^{r} \delta_i X_{t-i}^2 + \eta_t - \sum_{j=1}^{q} \beta_j \eta_{t-j},
\]

where \( r = \max(p,q) \) and \( \delta_i = \alpha_i + \beta_i \).

Hence find \( E(X_t^2) \) and explain why \( \sum_{i=1}^{r} \delta_i < 1 \).

ii. Let \( \sigma_{t,l}^2 \) be the l-step-ahead forecast function (from the time origin \( t \)) of \( \sigma_{t+l}^2 \). For a GARCH\((1,1)\) process, show that \( \sigma_{t,l}^2 \) is (recursively) generated by

\[
\sigma_{t,l}^2 = \alpha_0 + (\alpha_1 + \beta_1) \sigma_{t,l-1}^2, \quad l \geq 2.
\]
3. [12 marks]

(a) Suppose that \( \{X_{1,t}\} \) and \( \{X_{2,t}\} \) are formed from

\[
X_{1,t} = 0.6X_{1,t-1} + 0.2X_{2,t-1} + Z_{1,t} + 0.5Z_{2,t-1},
\]

\[
X_{2,t} = 0.4X_{1,t-1} + 0.4X_{2,t-1} + Z_{2,t} + 0.6Z_{2,t-1},
\]

where \( \{Z_{i,t}\} \sim WN(0,1) \) for \( i = 1, 2 \) and for all \( t \).

(i) Show that \( X_t = (X_{1,t}, X_{2,t})^T \) may be expressed as \( X_t - \Phi X_{t-1} = Z_t - \Theta Z_{t-1} \) for suitably chosen \( 2 \times 2 \) matrices \( \Phi \) and \( \Theta \), where \( Z_t = (Z_{1,t}, Z_{2,t})^T \) (\( T \) stands for transpose operation).

(ii) Prove that the above bivariate ARMA(1,1) process is stationary.

(iii) Suppose that a bivariate time series of 70 observations can be modelled by the above model. Given that \( \hat{X}_{70} = \hat{X}_{69,1} = (2.1, 3.2)^T \) and \( x_{70} = (3.8, 4.3)^T \), find an estimate of \( X_{71} \) (i.e. \( \hat{X}_{70,1} \)).

(b) Let \( \{X_t; t \in \mathbb{Z}\} \) be a stationary (real valued) bivariate process satisfying

\[
X_t = \phi X_{t-1} + U_t,
\]

where

\[
X_t = \begin{pmatrix} X_{1t} \\ X_{2t} \end{pmatrix}; \quad \phi = \begin{pmatrix} \alpha & 0 \\ \beta & 0 \end{pmatrix}; \quad U_t = \begin{pmatrix} \eta_t \\ \xi_t \end{pmatrix}
\]

with \( E(U_t) = 0 \) and \( \text{Cov}(U_t) = \text{diag}(\sigma^2, \tau^2) \).

Show that the cross spectrum of \( \{X_{1t}\} \) and \( \{X_{2t}\} \) is given by

\[
h_{X_1,X_2}(\omega) = \frac{\beta \sigma^2 e^{i\omega}}{2\pi(1 - 2\alpha \cos \omega + \alpha^2)}; \quad -\pi < \omega < \pi.
\]
4. [13 marks]

(a) A model for monthly observations \{X_t\} for a time series are found to be well fitted by an ARIMA(1, 0, 2)(1, 1, 1)_{12} model of the form

\[(I - \phi B)(I - \phi_{12} B^{12})(I - B^{12})X_t = (I - \theta_1 B - \theta_2 B^2)(I - \theta_{12} B^{12})Z_t,\]

where \{Z_t\} ~ WN(0, \sigma^2).

(i) Let \(Y_t = (I - B^{12})X_t\). State the conditions for stationarity and invertibility of \{Y_t\}.

(ii) Suppose that \(U_t = (I - \phi B)(I - \phi_{12} B^{12})(I - B^{12})X_t\). Find the theoretical autocorrelation function at lag \(k\) of \{U_t\}, \(\rho_U(k)\), for \(k = 1\) and \(k \geq 14\).

(iii) Find the \(\ell\)-step-ahead forecast function \(U_{t,\ell}\) for \(\ell = 1\) and \(\ell \geq 14\) based on the past \(X_t, X_{t-1}, \ldots\).

(b) Consider the general transfer function noise model with delay "b" given by

\[Y_t = \frac{\omega(B)B^b}{\delta(B)}X_t + N_t,\]

where \{\(N_t\)\} ~ WN(0, \sigma^2) and is uncorrelated with \{\(X_t\)\} for all \(t\); \(\omega(B) = \omega_0 - \omega_1 B \cdots - \omega_s B^s\); and \(\delta(B) = I - \delta_1 B \cdots - \delta_r B^r\).

Let \{\(X_t\)\} be an ARMA(p, q) process satisfying \(\phi(B)X_t = \theta(B)\alpha_t\), where \(\alpha_t ~ WN(0, \sigma^2_\alpha)\). Let \(\beta_t = \theta^{-1}(B)\phi(B)Y_t\).

Show that for a suitable choice of \(N_t^*\), \(\beta_t\) is given by

\[\beta_t = V(B)\alpha_t + N_t^*,\]

where \(V(B) = \frac{\omega(B)B^b}{\delta(B)} = \sum_{j=0}^{\infty} V_j B^j\).

Show that \{\(N_t^*\)\} and \{\(\alpha_t\)\} are uncorrelated. Find \(\text{Cov}(\beta_t, \alpha_{t-\kappa})\) and deduce that \(V_\kappa\) can be expressed as

\[V_\kappa = \frac{\sigma_\beta}{\sigma_\alpha} \rho_\beta_\alpha(\kappa),\]

where \(\rho_\beta_\alpha(\kappa)\) is the cross correlation function at lag \(\kappa\) of \{\(\beta_t\)\} and \{\(\alpha_t\)\}.
5. [20 marks]

(a) Let \((I - B)^d = \sum_{j=0}^{\infty} \pi_j B^j\), where \(B\) is the backshift operator such that \(B^j X_t = X_{t-j}, j \geq 0\) and \(I = B^0\) and \(d\) is a real number. It can be shown that \(\pi_j = \frac{\Gamma(j-d)}{\Gamma(j+1)\Gamma(-d)}\) and for large \(j\), \(\pi_j \sim \frac{j^{-d}}{\Gamma(-d)}\).

Suppose that \(\{X_t\}\) is an ARIMA\((0,d,0)\) process generated by \((I - B)^d X_t = Z_t\), where \(\{Z_t\} \sim WN(0, \sigma^2)\). Find constants \(\psi_j\) such that \(X_t = \sum_{j=0}^{\infty} \psi_j Z_{t-j}\) in terms of gamma functions, \(\Gamma(\cdot)\). Hence show that if \(-0.5 < d < 0.5\), then the process \(\{X_t\}\) is stationary and invertible. Write down the spectrum, \(f_X(\omega)\) of \(\{X_t\}\) and describe its behaviour near \(\omega \approx 0\) for \(-0.5 < d < 0\) and \(0 < d < 0.5\). (The case of \(d = 0\) is trivial.)

(b) Let \(I X(\omega_j)\) be the sample periodogram estimate of \(f_X(\omega)\) at \(\omega_j = \frac{2\pi j}{n} \in (0, \pi)\), where \(n\) denotes the number of observations. Under certain regularity conditions and for \(-0.5 < d < 0\), it is known that \(I X(\omega_j), j = 1, \cdots, [n/2] \) (\(n\) is odd and \([n/2]\) denotes the integer part of \(n/2\)), are asymptotically independent with distributions given by \(I X(\omega_j) \sim 0.5 f_X(\omega_j) \chi^2_2\).

Show that the sequence \(-ln\{\frac{f_X(\omega_j)}{f_X(\omega)}\}\), \(\omega_j \in (0, \pi)\) follows independent Gumbel distributions with mean 0.577216 and variance \(\frac{\pi^2}{6}\). [Hint: \(U\) is said to be a Gumbel distribution with parameters \(\alpha\) and \(\beta\) if the cdf of \(U\) is \(\exp\{-e^{-(u-\alpha)/\beta}\}\). In this case \(E(U) = \alpha + \beta \delta\) and \(Var(U) = \pi^2 \beta/6\), where \(\delta = 0.577216\).]

By considering \(ln\{f_X(\omega)\}\) near \(\omega \approx 0\), show that the associated estimating equation for estimating \(d\) can be written as

\[Y_j = a + bx_j + \epsilon_j, j = 1, \cdots, m \quad (m = n^\theta, 0 < \theta < 1),\]

where \(Y_j = ln\{I X(\omega_j)\}, x_j = ln\{4sin^2(\omega_j/2)\}, a = ln\{f_Z(0)\}, \epsilon_j = ln\{\frac{I X(\omega_j)}{I X(\omega)}\}\) and \(b = -d\).

Write down the least squares estimator \(\hat{b}\) of \(b\). Find \(E(\hat{b})\) and \(var(\hat{b})\).