SCHOOL OF MATHEMATICS AND STATISTICS

MATHEMATICAL STATISTICS, F07

STAT4 (TSA) - ADVANCED TIME SERIES ANALYSIS AND FORECASTING

— Semester 2, 2008 —

Assignment 2 - Due on 25 September

1. Suppose that $\{X_t\}$ is an ARIMA(0,d,0) process generated by $(I-B)^d X_t = Z_t$, where $\{Z_t\} \sim WN(0, \sigma^2)$.

   (a) Find constants ψ_j such that $X_t = \sum_{j=0}^{\infty} \psi_j Z_{t-j}$ in terms of gamma functions, $\Gamma(.)$. Hence show that if $d < 0.5$, then the process $\{X_t\}$ is stationary.

   (b) *Write down* the spectrum, $f_X(\omega)$ of $\{X_t\}$ and describe its behaviour near $\omega \approx 0$ for $0 < d < 0.5$. *Deduce* that $\{X_t\}$ has long memory when $0 < d < 0.5$.

   (c) Let $\rho_X(k)$ be the autocorrelation at lag k. *Given that* $\rho_X(k) = \frac{\Gamma(k+d)\Gamma(1-d)}{\Gamma(k-d+1)\Gamma(d)}$, *describe* the behaviour of $\rho_X(k)$ when $0 < d < 0.5$ as $k \to \infty$. *Deduce that* $\sum_{k=0}^{\infty} \rho_X(k) = \infty$.

2. For ARIMA(0,d,0) proces given in (1) show that for $-0.5 < d < 0$ $\sum_{k=0}^{\infty} \rho_X(k) < \infty$. In this case it can be shown that $I_X(\omega_j)$ are asymptotically independent and

$$I_X(\omega_j) \sim 0.5 f_X(\omega_j) \chi_2^2,$$

   where $I_X(\omega_j)$ be the sample periodogram estimate of $f_X(\omega_j)$ at $\omega_j = \frac{2\pi j}{n} \in (o, \pi)$, and n denotes the sample size.

   (a) *Show that* the sequence $\{-ln\frac{I_X(\omega_j)}{f_X(\omega_j)}\}$, $\omega_j \in (o, \pi)$ follows independent Gumbel distributions with mean 0.577216 and variance $\frac{\pi^2}{6}$. *[Hint: U is said to be a Gumbel distribution with parameters α and β if the cdf of U is* $exp\{-e^{-(u-\alpha)/\beta}\}$. *In this case* $E(U) = \alpha + \beta\delta$ *and* $Var(U) = \pi^2\beta/6$, *where* $\delta = 0.577216$.*]*

   (b) By considering $ln\{f_X(\omega)\}$ near $\omega \approx 0$, it can be shown that the associated estimating equation for estimating d can be written as

$$Y_j = a + bx_j + \epsilon_j, j = 1, \cdots, m \ (m = n^\theta, 0 < \theta < 1),$$

   where $Y_j = ln\{I_X(\omega_j)\}, x_j = ln\{4sin^2(\omega_j/2)\},\ a = ln\{f_Z(0)\},\ \epsilon_j = ln\{\frac{I_X(\omega_j)}{f_X(\omega_j)}\}$ and $b = -d$.

   Write down the least squares estimate $\hat{b}$ of b and *find* the mean and variance of $\hat{b}$. *Provide* a 95% CI for d assuming normality.

PTO for Ass3

Assignment 3 - Due on 9 October

1. A stationary process $\{X_t\}$ is said to be a GARCH(p,q) process if it satisfies

$$X_t = \sigma_t \epsilon_t, \ \sigma_t^2 = \alpha_0 + \sum_{i=1}^{p} \alpha_i X_{t-i}^2 + \sum_{j=1}^{q} \beta_j \sigma_{t-j}^2,$$

   where $\alpha_0 > 0, \ \alpha_i \geq 0, \ \beta_j \geq 0$ and $\{\epsilon_t\}$ is a sequence of iid random variables with mean zero and variance 1.

   (a) Given that $\eta_t = X_t^2 - \sigma_t^2$ is a martingale difference series *find* the value of r (in terms of p and q) and the constants δ_i (in terms of α_i's and β_j's) such that X_t^2 is an ARMA (r,q) process satisfying

$$X_t^2 = \alpha_0 + \sum_{i=1}^{r} \delta_i X_{t-i}^2 + \eta_t - \sum_{j=1}^{q} \beta_j \eta_{t-j}.$$

   (b) *Find* $E(X_t^2)$ and *explain why* $\sum_{i=1}^{r} \delta_i < 1$.

2. Consider the GARCH (1,1) process (by setting $p = q = 1$ in (1)) given by

$$X_t = \sigma_t \epsilon_t, \ \sigma_t^2 = \alpha_0 + \alpha X_{t-1}^2 + \beta \sigma_{t-1}^2,$$

   where $\{\epsilon_t\} \sim NID(0, 1)$ and $\alpha_0 > 0, \ \alpha > 0, \ \beta > 0$.

   (a) *Write down* the above GARCH (1,1) as ARMA (r,q) model for X_t^2 for suitable values of r and q.

   (b) *State* the conditions for second-order stationarity of GARCH (1,1).

   (c) *Show that* $Var(\eta_t) = \frac{2}{3} E(X_t^4)$.

   (d) *Find* the constants μ^* and ψ_j such that

$$X_t^2 = \mu^* + \sum_{j=0}^{\infty} \psi_j \eta_{t-j}.$$

   (e) Let $K^{(g)}$ be the GARCH kurtosis or the excess kurtosis of X_t given by $K^{(g)} = K^{(X)} - 3$. *Show that*

$$K^{(g)} = \frac{6a}{1 - 2a},$$

   where $a = \sum_{j=1}^{\infty} \psi_j^2$.

End of Ass3

NOTE: Technical Report will due on 30 October