1. Find the radius of convergence of the following series.

\[
\begin{align*}
(i) \quad & \sum_{n=1}^{\infty} 5^n x^n \\
(ii) \quad & \sum_{n=0}^{\infty} \frac{5^n x^n}{n^2} \\
(iii) \quad & \sum_{n=0}^{\infty} \frac{5^n x^n}{n!} \\
(iv) \quad & \sum_{n=1}^{\infty} \frac{5^n x^n}{n^n} \\
(v) \quad & \sum_{n=0}^{\infty} \frac{n x^n}{5^n} \\
(vi) \quad & \sum_{n=2}^{\infty} \frac{x^n}{5^n \ln n}.
\end{align*}
\]

Solution.

\( (i) \) The series \( \sum_{n=0}^{\infty} 5^n x^n = \sum_{n=0}^{\infty} (5x)^n \) is geometric, with \( r = 5x \) and so it converges for \( |5x| < 1 \) and diverges for \( |5x| > 1 \). Hence the radius of convergence is \( R = \frac{1}{5} \).

When \( x = \pm \frac{1}{5} \), the absolute value of the \( n \)-th term of the series is 1 for every \( n \) and so does not approach 0 as \( n \to \infty \). Hence the series diverges at each point \( \pm \frac{1}{5} \).

\( (ii) \) Use the ratio test.

\[
\left| \frac{a_{n+1}}{a_n} \right| = \left| \frac{\frac{5^{n+1} x^{n+1}}{(n+1)^2} n^2}{5^n x^n} \right| = |5x| \left( \frac{n}{n+1} \right)^2 \to |5x|, \text{ as } n \to \infty.
\]

Hence if \( |5x| < 1 \), then the series converges and if \( |5x| > 1 \) it diverges.

Thus the radius of convergence is \( R = \frac{1}{5} \).

When \( x = \pm \frac{1}{5} \), the series is \( \sum_{n=1}^{\infty} \pm \frac{1}{n^2} \) both of which converge absolutely, by comparison with the \( 2 \)-series.

\( (iii) \) Use the ratio test.

\[
\left| \frac{a_{n+1}}{a_n} \right| = \left| \frac{\frac{5^{n+1} x^{n+1}}{(n+1)!} n!}{5^n x^n} \right| = |5x| \frac{1}{n+1} \to 0, \text{ as } n \to \infty.
\]

Hence the series converges for every \( x \) and so the radius of convergence of is \( R = \infty \) in this case.
(iv) Use the ratio test.

\[
\left| \frac{a_{n+1}}{a_n} \right| = \left| \frac{5^{n+1} x^{n+1} n^n}{(n+1)^{n+1} 5^n x^n} \right| = \left| 5x \left( \frac{1}{1 + \frac{1}{n}} \right)^n \frac{1}{n+1} \right| \to \frac{5x}{e} 0 \text{ as } n \to \infty.
\]

Hence the series converges for every \( x \) and the radius of convergence is again \( R = \infty \).

(v) Use the ratio test.

\[
\left| \frac{a_{n+1}}{a_n} \right| = \left| \frac{(n+1)x^{n+1} 5^n}{5(n+1) n x^n} \right| = \left| \frac{x}{5} \right| \frac{n+1}{n} \to \frac{|x|}{5} \text{ as } n \to \infty.
\]

Hence the series converges if \( |x| < 5 \) and diverges if \( |x| > 5 \).

The radius of convergence is thus \( R = 5 \).

When \( x = 5 \), the \( n\)-th term \( a_n = \pm n \), and so does not approach 0 as \( n \to \infty \).
Hence the series diverges if \( x \pm 5 \).

(vi) Use the ratio test.

\[
\left| \frac{a_{n+1}}{a_n} \right| = \left| \frac{x^{n+1}}{5^{n+1} \ln(n+1)} \frac{5^n \ln n}{x^n} \right| = \left| \frac{x}{5} \right| \ln \frac{n}{\ln(n+1)} \to \frac{|x|}{5}, \text{ as } n \to \infty.
\]

Hence the series converges if \( |x| < 5 \) and diverges if \( |x| > 5 \). Hence the radius of convergence is \( R = 5 \).

When \( x = 5 \) the series is \( \sum_{n=2}^{\infty} \frac{1}{n \ln n} \) which diverges by comparison with \( \sum_{n=2}^{\infty} \frac{1}{n} \).

When \( x = -\frac{1}{5} \) the series becomes \( \sum_{n=2}^{\infty} (-1)^n \frac{1}{n \ln n} \) which converges (conditionally) by the alternating series test.

2. (i) Suppose that \( \sum_{n=0}^{\infty} a_n x^n \) has radius of convergence \( R \) and that \( |b_n| < |a_n| \), for all \( n \geq 0 \). Show that \( \sum_{n=0}^{\infty} b_n x^n \) has radius of convergence at least \( R \).

(ii) Give examples which show that the radius of convergence of \( \sum b_n x^n \) may be exactly \( R \) or strictly greater than \( R \).
Solution.

(i) If \( \sum_{n=0}^{\infty} a_n x^n \) has radius of convergence \( R \), then it converges absolutely for all \( x \) with \( |x| < R \). That is \( \sum_{n=0}^{\infty} |a_n x^n| \) converges, for all \( x \) with \( |x| < R \).

If \( |b_n| < |a_n| \), the \( |b_n x^n| < |a_n x^n| \) and so \( \sum_{n=0}^{\infty} |b_n x^n| \) converges by comparison with \( \sum_{n=0}^{\infty} |a_n x^n| \). That is for all \( x \) with \( |x| < R \), \( \sum_{n=0}^{\infty} b_n x^n \) converges absolutely.

(ii) The series \( \sum_{n=0}^{\infty} x^n \) has radius of convergence 1 while the series \( \sum_{n=0}^{\infty} \frac{x^n}{n!} \) also has radius of convergence 1. This shows that if \( |b_n| = \frac{1}{n!} \leq |a_n| = 1 \), the radius of convergence may not decrease.

On the other hand, if \( b_n = \frac{1}{n!} \) and \( a_n = 1 \), then \( |b_n| \leq |a_n| \), but the radius of convergence of \( \sum_{n=0}^{\infty} b_n \) is \( R = \infty \) while that of \( \sum_{n=0}^{\infty} x^n \) has \( R = 1 \). Hence the radius of convergence can definitely increase.

3. Find the interval of convergence of the series

(\( \text{(i)} \)) \( \sum_{n=0}^{\infty} \frac{(x-2)^n}{n!} \), \( \text{(ii)} \) \( \sum_{n=0}^{\infty} \frac{(x+5)^n}{n(n+1)(n+2)} \), \( \text{(iii)} \) \( \sum_{n=0}^{\infty} \frac{(2x+5)^n}{n^2(n+1)} \).

Solution.

(\( \text{(i)} \)) Let \( y = x - 2 \). The series is then \( \sum_{n=0}^{\infty} \frac{y^n}{n!} \) which converges for every \( y \). Hence the series \( \sum_{n=0}^{\infty} \frac{(x-2)^n}{n!} \) converges for every \( x \). Hence the interval of convergence in this case is the whole real line \((-\infty, \infty)\).

(\( \text{(ii)} \)) Use the ratio test.

\[
\left| \frac{a_{n+1}}{a_n} \right| = \left| \frac{(x+5)^{n+1}}{(n+1)(n+2)(n+3)} \cdot \frac{n(n+1)(n+2)}{(x+5)^n} \right| = |x+5| \frac{n}{n+3} \rightarrow |x+5| \text{ as } n \rightarrow \infty.
\]

So the series converges if \( |x+5| < 1 \) and diverges if \( |x+5| > 1 \). Hence the series has radius of convergence 1.

When \( x+5 = 1 \), that is when \( x = -4 \), the series is \( \sum_{n=0}^{\infty} \frac{1}{n(n+1)(n+2)} \) which converges by comparison with the 3-series.

If \( x+5 = -1 \), that is, if \( x = -6 \) the series becomes \( \sum_{n=0}^{\infty} \frac{1}{n(n+1)(n+2)} \) which converges absolutely by comparison with the 3-series. Hence the series has interval of convergence \([-6, -4]\).
Hence the series converges for every \( x \) in the interval \([-6, -4]\) and diverges otherwise.

(iii) Use the ratio test.

\[
\left| \frac{a_{n+1}}{a_n} \right| = \left| \frac{(2x + 5)^{n+1}}{(n+1)^2(2x+5)^n} \right|^{\frac{1}{n}} = \left| \frac{2x + 5}{n+1} \right|
\]

Hence the series converges if \( |2x + 5| < 1 \) and diverges if \( |2x + 5| > 1 \), that is, if \( |x + \frac{5}{2}| < \frac{1}{2} \) the series converges while if \( |x + \frac{5}{2}| > \frac{1}{2} \), the series diverges. Hence the series has radius of convergence \( R = \frac{1}{2} \).

If \( 2x + 5 = 1 \) the series is \( \sum_{n=1}^{\infty} \frac{1}{n^2(n+1)} \), which converges by comparison with the 3-series while if \( 2x + 5 = -1 \), the series is \( \sum_{n=1}^{\infty} (-1)^n \frac{1}{n^2(n+1)} \) which converges absolutely by comparison again with the 3-series.

So the series has interval of convergence \([-3, -2]\).

4. (i) Suppose that \( \{c_n\} = \{a_n + ib_n\} \), where \( a, b \in \mathbb{R} \), is a sequence of complex numbers such that \( \sum c_n \) converges absolutely. that both \( \sum a_n \) and \( \sum b_n \) are absolutely convergent real series.

(ii) Conversely, show that if \( \{a_n\} \), \( \{b_n\} \) are sequences of real numbers such that \( \sum a_n \) and \( \sum b_n \) are absolutely convergent, then \( \sum c_n = \sum (a_n + ib_n) \) is convergent.

Solution.

(i) Since \( \sum_{n=0}^{\infty} c_n \) is absolutely convergent, it follows that \( \sum_{n=0}^{\infty} |c_n| = \sum_{n=0}^{\infty} \sqrt{a_n^2 + b_n^2} \) converges. Now for every \( n \),

\[
|a_n|, |b_n| \leq \sqrt{a_n^2 + b_n^2}
\]

and so both \( \sum_{n=0}^{\infty} |a_n| \) and \( \sum_{n=0}^{\infty} |b_n| \) both converge.

Hence \( \sum_{n=0}^{\infty} a_n \) and \( \sum_{n=0}^{\infty} b_n \) are both absolutely convergent.

(ii) Since \( \sum_{n=0}^{\infty} a_n \) and \( \sum_{n=0}^{\infty} b_n \) are absolutely convergent series of real numbers and so are themselves convergent, it follows that

\[
\sum_{n=0}^{\infty} c_n = \sum_{n=0}^{\infty} (a_n + ib_n) = \sum_{n=0}^{\infty} a_n + i\sum_{n=0}^{\infty} b_n
\]

is a sum of two convergent series and so converges. (We are using here that we can rearrange the terms of an absolutely convergent series without affecting its sum.)
5. Find the radius of convergence of the series

\( (i) \sum_{n=0}^{\infty} z^n/n! \),

\( (ii) \sum_{n=1}^{\infty} (z+i)^n/n^2 \),

\( (iii) \sum_{n=1}^{\infty} \frac{(2z-i)^n}{n(n+1)(n+2)(n+3)} \).

Solution.

\( (i) \) This is a geometric series with \( r = \frac{z}{3} \) and so it converges if \( |z| < 3 \) and diverges if \( |z| > 3 \).

If \( |z| = 3 \), then the \( n \)-th term of the series has absolute value 1 for every \( n \) and so cannot tend to 0 as \( n \to \infty \). Hence the series The series has radius of convergence \( R = 3 \) and converges only at this points \( z \) for which \( |z| < 3 \).

\( (ii) \) Use the ratio test.

\[
\left| \frac{a_{n+1}}{a_n} \right| = \left| \frac{(z+i)^{n+1}}{(n+1)^2} \cdot \frac{n^2}{(z+i)^n} \right| \\
= |z+i| \cdot \frac{n^2}{(n+1)^2} \\
\to |z+i| \text{ as } n \to \infty .
\]

Hence if \( |z+i| < 1 \), the series converges, while if \( |z+i| > 1 \), it diverges.

Hence the radius of convergence is \( R = 1 \).

If \( |z+i| = 1 \), the series of absolute values is then \( \sum |c_n| = \sum_{n=1}^{\infty} \frac{1}{n^2(n+1)} \) and this converges by comparison with the 3–series. Hence the series \( \sum_{n=1}^{\infty} \frac{(z+i)^n}{n^2} \) converges for every \( z \) with \( |z+i| \leq 1 \).

\( (iii) \) Use the ratio test.

\[
\left| \frac{a_{n+1}}{a_n} \right| = \left| \frac{(2z-i)^{n+1}}{(n+1)(n+2)(n+3)} \cdot \frac{n(n+1)(n+2)}{(2z-i)^n} \right| \\
= |2z-i| \cdot \frac{n}{n+3} \\
\to |2z-i| \text{ as } n \to \infty .
\]

Hence the series converges if \( |2z-i| < 1 \) and diverges if \( |2z-i| > 1 \). Hence the series converges in the disc \( |z-\frac{i}{2}| < \frac{1}{2} \) and diverges when \( |z-\frac{i}{2}| > \frac{1}{2} \). It has radius of convergence \( R = \frac{1}{2} \).

When \( |z-\frac{i}{2}| = \frac{1}{2} \), the series of absolute values \( \sum_{n=1}^{\infty} \frac{(2z-i)^n}{n(n+1)(n+2)} \) converges by comparison with the 3–series. Hence the series \( \sum_{n=1}^{\infty} \frac{(2z-i)^n}{n(n+1)(n+2)} \) converges absolutely for every \( z \) with \( |z-\frac{i}{2}| \leq \frac{1}{2} \). This is a disc of radius \( \frac{1}{\sqrt{2}} \) with centre \( \frac{i}{2} \).