1. Prove that for all real numbers $x, y$,
   (a) $-|x| \leq x \leq |x|
   (b) $|x + y| \leq |x| + |y|,
   (c) $||x| - |y|| \leq |x - y|.$

**Solution**

(a) If $x \geq 0$, then $|x| = x$ and so $-|x| \leq 0 \leq x = |x|.$
If $x < 0$, then $|x| = -x$ and so $-|x| = x < 0 < |x|.$
Hence $-|x| \leq x \leq |x|$, for every $x$.

(b) Hence

$$-|x| \leq x \leq |x|$$
$$-|y| \leq y \leq |y|,$$

and adding these three inequalities, we get

$$-(|x| + |y|) \leq x + y \leq |x| + |y|.$$  

Hence

$$|x + y| \leq |x| + |y|.$$

(c) Now

$$|x| = |x - y + y| \leq |x - y| + |y|$$

and so

$$|x - y| \geq |x| - |y|.$$  

Similarly,

$$|y| = |y - x + x| \leq |y - x| + |x|$$

and so

$$|y - x| = |x - y| \geq |y| - |x|.$$  

Hence $|x - y|$ is greater than or equal to both $|x| - |y|$ and $-(|x| - |y|)$ and so

$$|x - y| \geq ||x| - |y||.$$

2. Suppose that $a_1 = 0, a_{n+1} = \sqrt{3 + a_n}$.
(a) Show that $\{a_n\}$ is a bounded sequence by showing that $0 \leq a_n < 5$ for all $n$.

(b) Show that if $n \geq 2,$

$$a_{n+1} - a_n = \frac{a_n - a_{n-1}}{\sqrt{3 + a_n + \sqrt{3 + a_{n-1}}} \leq \frac{a_n - a_{n-1}}{2\sqrt{3}} < \frac{a_n - a_{n-1}}{3}.}$$

(c) Now show that $a_{n+1} - a_n \leq \frac{1}{3^{n-1}}\sqrt{3}$ and hence that $\{a_n\}$ is a Cauchy sequence.

(d) Find $\lim a_n$ as $n \to \infty$.
Solution

(a) First $a_n$ is clearly non-negative for every $n$ and so $a_n \geq 0$ for all $n$. We'll prove by induction on $n$ that $a_n < 5$ for all $n \geq 0$.

First $a_1 = 0 < 5$.

Suppose that $a_n < 5$. Then $a_{n+1} = \sqrt{3 + a_n} < \sqrt{3 + 5} = \sqrt{8} < 5$. Hence $a_n < 5$ for all $n$ by induction and so $\{a_n\}$ is a bounded sequence.

Actually it is just as easy to show that $a_n \leq 3$, for every $n$. I wanted you to see that it is not necessary for you to pick the best possible bound ... any bound is good enough to show that the sequence is bounded.

(b) $a_{n+1} - a_n = \sqrt{3 + a_n} - \sqrt{3 + a_{n-1}}$

$= \frac{(\sqrt{3 + a_n} - \sqrt{3 + a_{n-1}})(\sqrt{3 + a_n} + \sqrt{3 + a_{n-1}})}{\sqrt{3 + a_n} + \sqrt{3 + a_{n-1}}}$

$= \frac{3 + a_n - 3 - a_{n-1}}{\sqrt{3 + a_n} + \sqrt{3 + a_{n-1}}}$

$= \frac{a_n - a_{n-1}}{\sqrt{3 + a_n} + \sqrt{3 + a_{n-1}}}$

$< \frac{a_n - a_{n-1}}{2\sqrt{3}}$

$< \frac{a_n - a_{n-1}}{3}$.

Hence

$a_{n+1} - a_n < \frac{a_n - a_{n-1}}{3}$

$< \frac{a_{n-1} - a_{n-2}}{3^2}$

$\vdots$

$< \frac{a_2 - a_1}{3^{n-1}}$

$= \frac{\sqrt{3}}{3^{n-1}}$.

Hence if $m > n$,

$|a_m - a_n| = |a_m - a_{m-1} + a_{m-1} - a_{m-2} + \ldots + a_{n+1} - a_n|$

$\leq |a_m - a_{m-1}| + |a_{m-1} - a_{m-2}| + \ldots + |a_{n+1} - a_n|$

$< \frac{\sqrt{3}}{3^{m-1}} + \ldots + \frac{\sqrt{3}}{3^{n-1}}$

$< \frac{\sqrt{3}}{3^{n-1}} \left( 1 + \frac{1}{3} + \frac{1}{3^2} + \ldots \right)$

$= \frac{\sqrt{3}}{3^{n-1}} \left( \frac{1}{1 - \frac{1}{3}} \right)$

$= \frac{3\sqrt{3}}{2.3^{n-1}}$

$< \frac{1}{3^{n-2}}$. 
Hence \( \{a_n\} \) is a Cauchy sequence because given \( \epsilon \), choose \( N > 2 + \frac{\ln \epsilon}{\ln 3} \). Then if \( m > n > N \), \( |a_m - a_n| < \frac{1}{3^{n-2}} < \epsilon \).

(c) As \( n \to \infty \) \( a_n \to \frac{1 + \sqrt{13}}{2} \), since if \( a_n \) is close to \( l \), \( a_{n+1} \) is also close to \( l \) and in the limit, we have \( l = \sqrt{3} + l \) and so

\[ l^2 - l - 3 = 0. \]

Hence \( l = \frac{1 \pm \sqrt{13}}{2} \). But the terms of the sequence are all positive and so the limit \( l \) can’t be \( \frac{1 - \sqrt{13}}{2} \).

3. Find the least upper bound and the greatest lower bound (if they exist) of the following sets. Also decide whether the LUBs and/or GLBs are elements of the sets.

(a) \( \{\frac{1}{n} : n \in \mathbb{N}^+\} \)  
(b) \( \{\frac{1}{n} : n \in \mathbb{Z}, n \neq 0\} \)

(c) \( \{x : x = 0, \text{ or } x = \frac{1}{n}, n \in \mathbb{N}^+\} \)  
(d) \( \{\frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \ldots, \frac{2}{3}, \frac{3}{4}, \frac{4}{5}, \ldots\} \).

**Solution**

(a) LUB 1, which is in the set; GLB 0, which is not in the set.

(b) The set is

\[ \{1, \frac{1}{2}, \frac{1}{3}, \ldots, -1, -\frac{1}{2}, -\frac{1}{3}, \ldots\}. \]

LUB 1 which is in the set; GLB -1 which is in the set.

(c) LUB 1, which is in the set; GLB 0, which is in the set.

(d) LUB 1 which is not in the set; GLB 0 which is not in the set.

4. Which if the following sequences are (a) bounded? (b) monotonic? Which are convergent?

(a) \( \{(0 \cdot 2)^n\} \)  
(b) \( \{\frac{3n + 1}{2n - 1}\} \)  
(c) \( \{\cos \frac{n\pi}{2}\} \)

(d) \( \{\frac{2^n}{100n}\} \)  
(e) \( \{\frac{5^n}{n!}\} \)  
(f) \( \{(1 + \frac{1}{n})^n : n \in \mathbb{N}^+\} \)

**Solution**

(a) Bounded above by 1 and below by 0; monotonic decreasing. Convergent with limit 0.

\[ a_{n+1} < a_n, \text{ since } \frac{a_{n+1}}{a_n} = 0.2 < 1. \]

(b) Bounded below by 0 and above by 4; monotonic decreasing. Convergent with limit \( \frac{3}{2} \).
\( a_{n+1} < a_n \) since
\[
\frac{a_{n+1}}{a_n} = \frac{3n + 42n - 1}{2n + 13n + 1} = \frac{6n^2 + 5n - 4}{6n^2 + 5n + 1} = 1 - \frac{5}{6n^2 + 5n + 1} < 1, \text{ if } n \geq 1.
\]

(c) The sequence is 
\[\{0, -1, 0, 1, 0, -1, 0, 1, \ldots\}.\]

(d) Bounded below by 0. Not bounded above. Not convergent.
\[
\frac{a_{n+1}}{a_n} = \frac{2^{n+1} 100n}{100(n+1) 2^n} = \frac{2n}{n+1} > 1, \text{ if } n \geq 2.
\]
Hence 
\[a_{n+1} > \frac{3}{2}a_n > \ldots > \left(\frac{3}{2}\right)^n a_1 \to \infty\]
as \( n \to \infty \).

(e) The sequence is 
\[\{5, 12.5, 20.83, 26.04, 26.04, 21.70, \ldots\}.\]
Bounded above by \(\frac{5^4}{24}\) and bounded below by 0. Obviously not monotonic but monotonic decreasing if \( n \geq 5 \). Convergent with limit 0.
\[
\frac{a_{n+1}}{a_n} = \frac{5^{n+1} n!}{(n+1)! 5^n} = \frac{5}{n+1} \leq 1, \text{ if } n \geq 5.
\]

(f) Bounded above by 3 and below by 0. Monotonic increasing and convergent with limit e.
\[
(1 + \frac{1}{n})^n = 1 + \frac{n}{n} + \frac{n(n-1)(n-2)}{2!n^2} + \frac{n(n-1)(n-2)(n-3)}{3!n^3} + \ldots.
\]
\[
= 1 + 1 + \frac{1}{2!}\left(1 - \frac{1}{n}\right) + \frac{1}{3!}\left(1 - \frac{1}{n}\right)\left(1 - \frac{2}{n}\right) + \ldots + \frac{1}{n!}\left(1 - \frac{1}{n}\right)\left(1 - \frac{2}{n}\right)\ldots\left(1 - \frac{n-1}{n}\right)
\]
\[
< 1 + 1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \ldots
\]
\[
= 1 + \frac{1}{1 - \frac{1}{2}} = 3.
\]
One way to see that the sequence is monotonic increasing is to notice that each term in
\[
(1 + \frac{1}{n})^n = 1 + \frac{n}{n} + \frac{n(n-1)}{2!n^2} + \frac{n(n-1)(n-2)}{3!n^3} + \ldots.
\]
This is positive and increases with $n$. Hence $(1 + \frac{1}{n})^n$ increases with $n$.

5. Answer true or false. If false, give reasons.

(a) Every monotonic decreasing sequence has a limit.
(b) $\{\sqrt{n}\}$ is a convergent sequence.
(c) $\{n - \sqrt{(n + 5)(n + 7)}\}$ is a divergent sequence.
(d) If $\{a_n\}$ is a monotonic decreasing sequence of positive real numbers, then $\lim_{n \to \infty} a_n$ is also positive.
(e) Every real number is a limit of a sequence of rational numbers.

Solution

(a) False. $\{0, -1, -2, \ldots\}$.
(b) True. (By L'Hôpital, $x^{1/x}$ tends to 1 as $x \to \infty$.
(c) False.
\[
n - \sqrt{(n + 5)(n + 7)} = \frac{(n - \sqrt{(n + 5)(n + 7)}) \left( n + \sqrt{(n + 5)(n + 7)} \right)}{n + \sqrt{(n + 5)(n + 7)}}
\]
\[
= \frac{n^2 - (n + 5)(n + 7)}{n + \sqrt{(n + 5)(n + 7)}}
\]
\[
= \frac{-12n - 35}{n + \sqrt{(n + 5)(n + 7)}}
\]
\[
= \frac{-12 - \frac{35}{n}}{1 + \frac{1}{n} \left( 1 + \frac{2}{n} \right)}
\]
\[
\to \frac{-12}{2} = -6.
\]
(d) False $\{1, \frac{1}{2}, \frac{1}{3}, \ldots\}$ is monotonic decreasing, positive with limit 0.
(e) True. Every real number is the limit of a decimal expansion and the truncated terms in each representation are all rational numbers.

6. (a) Prove that given any two rational numbers, $a$ and $b$, there is a rational number $c$ with $a < c < b$.

(b) Prove that given any two rational numbers $a$ and $b$, there exists an irrational number $d$ such that $a < d < b$.

Solution

(a) If $a < b$ are rational numbers, then
\[
a < \frac{a + b}{2} < b
\]
and $\frac{a + b}{2}$ are rational.
(b) Suppose that \( a < b \) where \( a \) and \( b \) are rational. Then choose \( k \in \mathbb{N} \) such that
\[
\frac{\sqrt{2}}{k} < \frac{b - a}{2}.
\]
Consider the irrational numbers \( x \frac{\sqrt{2}}{k} \), where \( x \) is an integer.

These are all irrational numbers which are \( \frac{\sqrt{2}}{k} \) apart. Hence at least one of them must lie in the interval \((a, b)\).

7. (a) Prove that if \( 0 < a < 2 \), then \( a < \sqrt{2a} < 2 \).

(b) Prove that the sequence \( \sqrt{2}, \sqrt{2\sqrt{2}}, \sqrt{2\sqrt{2\sqrt{2}}}, \ldots \) converges.

(c) Find its limit. (Hint: Use the fact that if \( \lim_{n \to \infty} a_n = l \), then \( \lim_{n \to \infty} \sqrt{2a_n} = \sqrt{2l} \)).

**Solution**

(a) \( a < \sqrt{2a} \) with \( a \) positive, if and only if \( a^2 < 2a \) which is true if \( a < 2 \). Similarly \( \sqrt{2a} < 2 \).

(b) The sequence is defined by \( a_1 = \sqrt{2}, a_{n+1} = \sqrt{2a_n} \). Clearly the sequence consists only of positive numbers and so the sequence is bounded below by 0. By part (i), we have \( a_n < 2 \) for every \( n \). Hence the sequence is bounded above.

Also
\[
a_{n+1} - a_n = \sqrt{2a_n} - a_n > 0, \text{ by part (i)}.
\]

Hence \( a_{n+1} > a_n \), for all \( n \geq 1 \) and the sequence is a monotonic increasing sequence. Hence the sequence has a limit \( l \).

(c) Since \( a_{n+1} = \sqrt{2a_n} \), in the limit, \( l = \sqrt{2l} \) and so \( l = 2 \).