1. Apply the Mean Value Theorem to $f(x) = \sqrt{x}$ on the interval $[64, 66]$ to show that $8 + \frac{1}{9} < \sqrt{66} < 8 + \frac{1}{8}$.

**Solution**

First $f(x) = \sqrt{x}$ is obviously a differentiable function on $[64, 66]$ and so the MVT applies to it. Hence

$$\frac{f(66) - f(64)}{2} = \frac{\sqrt{66} - 8}{2} = f'(c),$$

where $64 < c < 66$.

Since $64 < c < 66$,

$$\frac{1}{9} < \frac{1}{\sqrt{66}} < \frac{1}{\sqrt{c}} < \frac{1}{\sqrt{64}} = \frac{1}{8}.$$

Hence

$$\frac{1}{9} < \sqrt{66} - 8 < \frac{1}{8}.$$

2. What is wrong with the following application of L'Hôpital’s rule?

$$\lim_{x \to 1} \frac{x^3 + x - 2}{x^2 - 3x + 2} = \lim_{x \to 1} \frac{3x^2 + 1}{2x - 3} = \lim_{x \to 1} \frac{6x}{2} = 3.$$

**Solution**

$$\lim_{x \to 1} \frac{3x^2 + 1}{2x - 3} = \lim_{x \to 1} \frac{3x^2 + 1}{2x - 3}$$

$$= \frac{4}{-1} = -4.$$

This is a direct application of the limit laws, since $\lim_{x \to 1}(2x - 3) = -1 \neq 0$.

The quotient involved in $\lim_{x \to 1}(3x^2 + 1)$ is not an indeterminate form.

3. Evaluate the following limits:
(a) \( \lim_{x \to 0} \frac{e^x - e^{-x}}{\sin 3x} \)

(b) \( \lim_{x \to 0} \frac{e^{2x} - 1 - 2x - 2x^2}{x^3} \)

(c) \( \lim_{x \to \infty} x^{-\frac{1}{100}} \ln x \)

(d) \( \lim_{x \to 0^+} x^{\frac{1}{100}} \ln x \).

**Solution**

(a)
\[
\lim_{x \to 0} \frac{e^x - e^{-x}}{\sin 3x} = \lim_{x \to 0} \frac{e^x + e^{-x}}{3 \cos 3x} = \frac{2}{3}.
\]

(b)
\[
\lim_{x \to 0} \frac{e^{2x} - 1 - 2x - 2x^2}{x^3} = \lim_{x \to 0} \frac{2e^{2x} - 2 - 4x}{3x^2}
= \lim_{x \to 0} \frac{4e^{2x} - 4}{6x}
= \lim_{x \to 0} \frac{8e^{2x}}{6}
= \frac{8}{6}.
\]

(c)
\[
\lim_{x \to \infty} x^{-\frac{1}{100}} \ln x = \lim_{x \to \infty} \frac{\ln x}{x^{\frac{1}{100}}}
= \lim_{x \to \infty} \frac{1}{\frac{1}{100} x^{\frac{99}{100}}}
= \lim_{x \to \infty} 100x^{\frac{1}{100}}
= 0.
\]

(d)
\[
\lim_{x \to 0^+} x^{\frac{1}{100}} \ln x = \lim_{x \to 0^+} \frac{\ln x}{x^{-\frac{1}{100}}}
= \lim_{x \to 0^+} \frac{\frac{1}{x}}{-100x^{\frac{101}{100}}}
= \lim_{x \to 0^+} -100x^{\frac{1}{100}}
= 0.
\]

4. For each of the following series, write down the first few terms in the sequence \( \{S_n\} \) of partial sums. Then test for convergence and find the sum, if it exists.

(a) \[\sum_{n=1}^{\infty} (-1)^n\]

(b) \[\sum_{n=1}^{\infty} (-1)^n n\]

(c) \[\sum_{n=1}^{\infty} (0.7)^n\]

(d) \[\sum_{n=1}^{\infty} \ln \left(\frac{n+1}{n}\right)\].

**Solution**

(a) \( S_1 = -1, \ S_2 = 0, \ S_3 = -1, \ S_4 = 0, \ldots \) This sequence does not converge.
(b) \(S_1 = -1, S_2 = -1 + 2 = 1, S_3 = 1 - 3 = -2, S_4 = -2 + 4 = 2, S_5 = 2 - 5 = -3, S_6 = -3 + 6 = 3, \ldots\) Hence \(S_{2n-1} = -n, S_{2n} = n.\) This sequence does not converge.

(c) 
\[
S_n = (0.7) + (0.7)^2 + (0.7)^3 + \ldots + (0.7)^n
\]
\[
= (0.7)(\frac{1 - 0.7^n}{1 - 0.7})
\]
\[
= \frac{0.7}{0.3} \text{ as } n \to \infty
\]
\[
= \frac{7}{3}.
\]

(d) 
\[
S_1 = \ln 2 - \ln 1 = \ln 2
\]
\[
S_2 = \ln 3 - \ln 2 + \ln 2 = \ln 3
\]
\[
S_3 = \ln 3 + \ln 4 - \ln 3 = \ln 4
\]
\[
\vdots
\]
Hence \(S_n = \ln(n+1)\) and this sequence tends to infinity as \(n \to \infty.\)

5. Decide whether \(\sum_{n=1}^{\infty} \frac{3^n - 2}{5^n}\) converges and find its sum.

Solution

\[
\sum_{n=1}^{\infty} \frac{3^n - 2}{5^n} = \sum_{n=1}^{\infty} \frac{3^n}{5^n} - 2 \sum_{n=1}^{\infty} \left(\frac{1}{5}\right)^n
\]
\[
= \frac{3}{5} - \frac{3}{5} \cdot \frac{1}{5} - 2 \cdot \frac{1}{5} - \frac{1}{5}
\]
\[
= \frac{3}{5} \cdot \frac{25}{54}
\]
\[
= 1.
\]

6. Write the following numbers as infinite series and hence write them as rational numbers:

(a) 0.123412341234 \ldots
(b) 0.249999 \ldots

Solution

(a)
\[
0.123412341234 \ldots = 1234 \cdot \frac{1}{10^4} + 1234 \cdot \frac{1}{10^8} + 1234 \cdot \frac{1}{10^{12}} + \ldots
\]
\[
= \frac{1234}{10^4} \cdot \frac{1}{1 - \frac{1}{10^4}}
\]
\[
= \frac{1234}{9999}.
\]
(b) 

\[ 0.249999 \ldots = \frac{24}{100} + \frac{9}{10^3} + \frac{9}{10^4} + \ldots \]

\[ = \frac{24}{100} + \frac{9}{10^3} \left(1 - \frac{1}{10}\right) \]

\[ = \frac{24}{100} + \frac{9}{10^3} \frac{9}{10} \]

\[ = \frac{24}{100} + \frac{1}{100} \]

\[ = \frac{25}{100} = 0.25. \]

7. True or false

(a) If the series \( \sum a_n \) converges, then the sequence \( \{a_n\} \) converges.

(b) If \( \sum a_n \) diverges, then the sequence \( \{a_n\} \) diverges.

(c) The series \( 1 + \frac{1}{3} + \frac{1}{5} + \frac{1}{7} + \ldots \) diverges.

(d) If \( \sum a_n = l \) and \( \sum b_n = m \) then \( \sum a_nb_n = lm. \)

(e) \( \sum (\frac{1}{n} - \frac{1}{n+1}) \) converges.

(f) \( \frac{1}{1} + \frac{1}{5} + \frac{1}{9} + \frac{1}{13} + \ldots \) is a convergent series.

(g) \( \sum_{n=1}^{\infty} \frac{n}{n^2 + 1} \) is a divergent series.

Solution

(a) True. \( a_n \to 0 \) as \( n \to \infty. \)

(b) False. \( \sum \frac{1}{n} \) diverges, while the sequence \( \{\frac{1}{n}\} \) converges to 0 as \( n \to \infty. \)

(c) True.

\[ 1 + \frac{1}{3} + \frac{1}{5} + \frac{1}{7} + \ldots > \frac{1}{2} + \frac{1}{4} + \ldots \]

\[ = \frac{1}{2} \left(1 + \frac{1}{2} + \frac{1}{3} + \ldots\right) \]

\[ > \frac{1}{2}K, \text{ for any number } K. \]

(d) False

\[ \sum a_n = 1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \ldots = \frac{1}{1 - \frac{1}{2}} = 2 \]

while

\[ \sum a_na_n = 1 + \frac{1}{4} + \frac{1}{16} + \frac{1}{64} + \ldots = \frac{1}{1 - \frac{1}{4}} = \frac{4}{3} \neq 4. \]

(e) True

\[ \sum \left(\frac{1}{n} - \frac{1}{n+1}\right) = 1 - \frac{1}{2} + \frac{1}{2} - \frac{1}{3} + \frac{1}{3} - \frac{1}{4} + \ldots = 1. \]
(f) False
\[
\frac{1}{1} + \frac{1}{5} + \frac{1}{9} + \frac{1}{13} + \ldots > \frac{1}{4} + \frac{1}{8} + \frac{1}{12} + \ldots
= \frac{1}{4}(1 + \frac{1}{2} + \frac{1}{3} + \ldots)
\]
> \frac{1}{4} K \text{ for any real number } K.

(g) True.
\[
\frac{n}{n^2 + 1} > \frac{n}{n^2 + n} = \frac{1}{2n}.
\]
Since
\[
\sum \frac{1}{2n} = \frac{1}{2} \sum \frac{1}{n}
\]
and diverges.

8. (Harder)
(a) Suppose that \( \{S_n\} \) is the sequence of partial sums of the series \( a_1 + a_2 + \ldots \). If \( S_n \to l \) and \( n \to \infty \), show that
\[
\frac{S_1 + S_2 + \ldots + S_n}{n} \to l \text{ when } n \to \infty.
\]
That is, if a series is convergent with sum \( l \) then the average of its partial sums also converges to \( l \).

(b) Show that if \( \{a_n\} = \{(−1)^{n−1}\} = \{1,−1,1,−1,\ldots \}, \) then the sequence of partial sums \( \{S_n\} = \{1,0,1,0,\ldots \} \) produces the ‘averaging’ sequence
\[
S_1 = 1, \quad \frac{S_1 + S_2}{2} = \frac{1}{2}, \quad \frac{S_1 + S_2 + S_3}{3} = \frac{2}{3}, \quad \frac{S_1 + S_2 + S_3 + S_4}{4} = \frac{2}{4}, \quad \frac{S_1 + S_2 + S_3 + S_4 + S_5}{5} = \frac{3}{5} \ldots.
\]
Show that this sequence has limit \( \frac{1}{2} \) and so we could consider the sum \( 1−1+1−1+\ldots \) to have ‘average’ sum \( \frac{1}{2} \).

\textbf{Solution}

(a) Since \( S_n \to l \) as \( n \to \infty \), then given \( \epsilon \), there exists a number \( N \) such that if \( n > N \),
\[
|S_n − l| < \frac{\epsilon}{2}.
\]
For the moment fix that number \( N \).

Since \( \{S_n\} \) is a convergent sequence, it is bounded and so there exists a real number \( K \) such that \( |S_n| \leq K \), for every \( n \). Then \( l \leq K \), because \( S_n \to l \) as \( n \to \infty \). Choose \( n > N \) such that
\[
\frac{NK}{m} < \frac{\epsilon}{4}.
\]
This is always possible since \( N \) is a fixed number and \( K \) is also constant. Then if \( m > n \),
\[
\left| \frac{S_1 + S_2 + \ldots + S_m}{m} − l \right| = \left| \frac{(S_1 + S_2 + \ldots + S_N) + S_{N+1} + \ldots + S_m}{m} − l \right|
\leq \left| \frac{S_1 + S_2 + \ldots + S_N}{m} \right| + \left| \frac{S_{N+1} − l}{m} \right| + \left| \frac{S_{N+2} − l}{m} \right| + \ldots + \left| \frac{S_m − l}{m} \right| + \left| \frac{−NI}{m} \right|
\leq \frac{NK}{m} + \frac{m − N}{m} \frac{\epsilon}{2} + \frac{|NI|}{m}
\leq \frac{\epsilon}{4} + \frac{\epsilon}{2} + \frac{\epsilon}{4}
= \epsilon.
\]
Hence
\[ \frac{S_1 + S_2 + \ldots + S_m}{m} \to l, \text{ as } m \to \infty. \]

(b) It is clear that
\[ \frac{S_1 + S_2 + \ldots + S_{2n}}{2n} = \frac{n}{2n} \]
while
\[ \frac{S_1 + S_2 + \ldots + S_{2n}}{2n + 1} = \frac{n}{2n + 1}. \]

Clearly this approaches \( \frac{1}{2} \) as \( n \to \infty \).