1. Prove that for all real numbers $x, y$,
   (a) $-|x| \leq x \leq |x|$
   (b) $|x + y| \leq |x| + |y|$, 
   (c) $||x| - |y|| \leq |x - y|$.

**Solution**
(a) If $x \geq 0$, then $|x| = x$ and so $-|x| \leq 0 \leq x = |x|$. If $x < 0$, then $|x| = -x$ and so $-|x| = x < 0 < |x|$. Hence $-|x| \leq x \leq |x|$, for every $x$.
(b) Hence
   
   $|y| = |x| \leq |x| + |y|$

   and adding these three inequalities, we get
   
   $-(|x| + |y|) \leq x + y \leq |x| + |y|$. 

   Hence
   
   $|x + y| \leq |x| + |y|$. 

(c) Now
   
   $|x| = |x - y + y| \leq |x - y| + |y|$

   and so
   
   $|x - y| \geq |x| - |y|$. 

   Similarly,
   
   $|y| = |y - x + x| \leq |y - x| + |x|$

   and so
   
   $|y - x| = |x - y| \geq |y| - |x|$. 

   Hence $|x - y|$ is greater than or equal to both $|x| - |y|$ and $-(|x| - |y|)$ and so
   
   $|x - y| \geq ||x| - |y||$. 

2. Find the least upper bound and the greatest lower bound (if they exist) of the following sets. Also decide whether the LUBs and/or GLBs are elements of the sets.

   (a) $\{\frac{1}{n} : n \in \mathbb{N}^+\}$  
   (b) $\{\frac{1}{n} : n \in \mathbb{Z}, n \neq 0\}$

   (c) $\{x : x = 0, \text{ or } x = \frac{1}{n}, n \in \mathbb{N}^+\}$  
   (d) $\{x : x^2 + x + 1 \geq 0\}$

   (e) $\{\frac{1}{n} + (-1)^n : n \in \mathbb{N}^+\}$  
   (f) $\{\frac{1}{2}, \frac{1}{3}, \ldots, \frac{2}{3}, \frac{3}{4}, \ldots\}$. 

**Solution**
(a) LUB 1, which is in the set; GLB 0, which is not in the set.
(b) The set is
\[ \{1, \frac{1}{2}, \frac{1}{3}, \ldots, -1, -\frac{1}{2}, -\frac{1}{3}, \ldots\} \]
LUB 1 which is in the set; GLB –1 which is in the set.
(c) LUB 1, which is in the set; GLB 0, which is in the set.
(d)
\[ x^2 + x + 1 = (x + \frac{1}{2})^2 + \frac{3}{4} \geq \frac{3}{4} \]
Hence the set is the set of all real numbers \( \mathbb{R} \) and so is neither bounded above or below.
(e) The set is
\[ \{0, \frac{3}{2}, \frac{2}{3}, \frac{5}{4}, \frac{7}{6}, \frac{6}{7}, \ldots\} \]
LUB \( \frac{3}{2} \) which is in the set; GLB –1 which is not in the set.
(f) LUB 1 which is not in the set; GLB 0 which is not in the set.

3. Which if the following sequences are (a) bounded? (b) monotonic? Which are convergent?

(a) \( \{0.2^n\} \)
(b) \( \{\frac{3n+1}{2n-1}\} \)
(c) \( \{\cos \frac{n\pi}{2}\} \)
(d) \( \{\frac{2^n}{100n}\} \)
(e) \( \{\frac{5^n}{n!}\} \)
(f) \( \{(1 + \frac{1}{n})^n : n \in \mathbb{N}^+\} \)

**Solution**

(a) Bounded above by 1 and below by 0; monotonic decreasing. Convergent with limit 0.
\[ a_{n+1} < a_n, \quad \text{since} \quad \frac{a_{n+1}}{a_n} = 0.2 < 1. \]

(b) Bounded below by 0 and above by 4; monotonic decreasing. Convergent with limit \( \frac{3}{2} \).
\[ a_{n+1} < a_n, \quad \text{since} \quad \frac{a_{n+1}}{a_n} = \frac{3n + 4n - 1}{2n + 1} = \frac{6n^2 + 5n - 4}{6n^2 + 5n + 1} = 1 - \frac{5}{6n^2 + 5n + 1} < 1, \quad \text{if} \quad n \geq 1. \]

(c) The sequence is
\[ \{0, -1, 0, 1, 0, -1, 0, 1, \ldots\}. \]

(d) Bounded below by 0. Not bounded above. Not monotonic, though it is monotonic increasing for \( n \geq 15 \). Not convergent.
\[ \frac{a_{n+1}}{a_n} = \frac{2^{n+1}}{100(n+1)} \cdot \frac{100n}{2^n} = \frac{2n}{n+1} > \frac{3}{2} > 1, \quad \text{if} \quad n \geq 3. \]
This is easy to prove by induction on $n$.
Hence
\[ a_{n+1} > \frac{3}{2} a_n > \ldots > \left(\frac{3}{2}\right)^n a_1 \to \infty \]
as $n \to \infty$.

(e) The sequence is
\[ \{5, 12.5, 20.83, 26.04, 26.04, 21.70, \ldots \} \]
Bounded above by $\frac{5^4}{24}$ and bounded below by 0. Obviously not monotonic but monotonic decreasing if $n \geq 5$. Convergent with limit 0.
\[ \frac{a_{n+1}}{a_n} = \frac{5^{n+1} n!}{(n+1)! 5^n} = \frac{5}{n+1} \leq 1, \text{ if } n \geq 5. \]

(f) Bounded above by 3 and below by 0. Monotonic increasing and convergent with limit $e$.
\[
\left(1 + \frac{1}{n}\right)^n = 1 + \frac{1}{n} + \frac{n(n-1)}{2! n^2} + \frac{n(n-1)(n-2)}{3! n^3} + \ldots
\]
\[
= 1 + 1 + \frac{1}{2!} (1 - \frac{1}{n}) + \frac{1}{3!} (1 - \frac{1}{n})(1 - \frac{2}{n}) + \ldots + \frac{1}{n!} (1 - \frac{1}{n})(1 - \frac{2}{n}) \ldots (1 - \frac{n-1}{n})
\]
\[
< 1 + 1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \ldots
\]
\[
= 1 + \frac{1}{1 - \frac{1}{2}}
\]
\[
= 3.
\]
One way to see that the sequence is monotonic increasing is to notice that each term in
\[
\left(1 + \frac{1}{n}\right)^n = 1 + \frac{n}{n} + \frac{n(n-1)}{2! n^2} + \frac{n(n-1)(n-2)}{3! n^3} + \ldots
\]
has the form
\[
\frac{1}{p!} (1 - \frac{1}{n})(1 - \frac{2}{n}) \ldots (1 - \frac{p-1}{n}).
\]
This is positive and increases with $n$. Hence $\left(1 + \frac{1}{n}\right)^n$ increases with $n$.

4. (a) Prove that given any two rational numbers, $a$ and $b$, there is a rational number $c$ with $a < c < b$.

(b) Prove that given any two rational numbers $a$ and $b$, there exists an irrational number $d$ such that $a < d < b$.

Solution

(a) If $a < b$ are rational numbers, then
\[ a < \frac{a+b}{2} < b \]
and $\frac{a+b}{2}$ is rational.
(b) Suppose that $a < b$ where $a$ and $b$ are rational. Then choose $k \in \mathbb{N}$ such that
\[
\frac{\sqrt{2}}{k} < \frac{b - a}{2}.
\]
Consider the irrational numbers $x \frac{\sqrt{2}}{k}$, where $x$ is an integer.
These are all irrational numbers which are $\frac{\sqrt{2}}{k}$ apart. Hence at least one of them must lie in the interval $(a, b)$.

5. Answer true or false. If false, give reasons.
(a) Every monotonic decreasing sequence has a limit.
(b) $\{\sqrt{n}\}$ is a convergent sequence.
(c) Between any two irrational numbers $a, b$ there is an irrational number $c$ with $a < c < b$.
(d) $\{n - \sqrt{n + 5\sqrt{n + 7}}\}$ is a divergent sequence.
(e) If $\{a_n\}$ is a monotonic decreasing sequence of positive real numbers, then $\lim_{n \to \infty} a_n$ is also positive.
(f) Every real number is a limit of a sequence of rational numbers.

**Solution**

(a) False. $\{0, -1, -2, \ldots\}$.
(b) True. (By L’Hôpital, $x^{1/x}$ tends to 1 as $x \to \infty$.
(c) True. The proof in Exercise (ii) doesn’t need $a$ and $b$ to be rational.
(d) False.
\[
n - \sqrt{n + 5\sqrt{n + 7}} = \frac{(n - \sqrt{n + 5\sqrt{n + 7}})(n + \sqrt{n + 5\sqrt{n + 7}})}{n + \sqrt{n + 5\sqrt{n + 7}}}
\]
\[
= \frac{n^2 - (n + 5)(n + 7)}{n + \sqrt{n + 5\sqrt{n + 7}}}
\]
\[
= \frac{n^2 - n^2 - 12n - 35}{n + \sqrt{n + 5\sqrt{n + 7}}}
\]
\[
= \frac{12 - \frac{35}{n}}{1 + \sqrt{1 + \frac{7}{n}}}
\]
\[
\to -12
\]
\[
= -\frac{2}{2}
\]
\[
= -6.
\]
(e) False $\{1, \frac{1}{2}, \frac{1}{3}, \ldots\}$ is monotonic decreasing, positive with limit 0.
(f) True. Every real number is the limit of a decimal expansion and the truncated terms in each representation are all rational numbers.

6. (Harder)
(a) Prove that if $0 < a < 2$, then $a < \sqrt{2a} < 2$.
(b) Prove that the sequence $\sqrt{2}, \sqrt{2\sqrt{2}}, \sqrt{2\sqrt{2\sqrt{2}}}, \ldots$ converges.
(c) Find its limit. (Hint: Use the fact that \( \lim_{n \to \infty} a_n = l \), then \( \lim_{n \to \infty} \sqrt{2a_n} = \sqrt{2l} \).

**Solution**

(a) \( a < \sqrt{2a} \) with a positive, if and only if \( a^2 < 2a \) which is true if \( a < 2 \). Similarly \( \sqrt{2a} < 2 \).

(b) The sequence is defined by \( a_1 = \sqrt{2}, a_{n+1} = \sqrt{2a_n} \). Part (i) then gives immediately that \( a_n < 2 \) for every \( n \). Hence the sequence is a Bounded sequence.

Also \[
\frac{a_{n+1}}{a_n} = \frac{\sqrt{2a_n}}{\sqrt{2a_{n-1}}} = \frac{a_n}{a_{n-1}}.
\]

Hence since \( a_1 = \sqrt{2} < a_2 = \sqrt{2\sqrt{2}} \), it is an easy induction to show that \( a_{n+1} > a_n \), for all \( n \geq 1 \). Hence the sequence is a monotonic increasing sequence.

Hence the sequence has a limit \( l \).

(c) Since \( a_{n+1} = \sqrt{2a_n} \), in the limit, \( l = \sqrt{2l} \) and so \( l = 2 \).

7. (Harder)

(a) Show that if \( x \) and \( y \) are positive real numbers then \( \sqrt{xy} \leq \frac{x + y}{2} \), with equality only when \( x = y \).

(b) Suppose that \( 0 < a_1 < b_1 \). Define

\[
a_{n+1} = \sqrt{a_n b_n} \quad b_{n+1} = \frac{a_n + b_n}{2}.
\]

(i) Show that the sequence \( \{a_n\} \) is monotonic increasing and bounded above, while \( \{b_n\} \) is monotonic decreasing and bounded below.

(ii) Prove that they have the same limit.

**Solution**

(a)

\[
\sqrt{xy} \leq \frac{x + y}{2} \text{ if and only if } xy \leq \frac{x^2 + 2xy + y^2}{4} \text{ if and only if }
\]

\[
4xy \leq x^2 + 2xy + y^2 \text{ if and only if } 0 \leq x^2 - 2xy + y^2 \text{ if and only if } 0 \leq (x - y)^2.
\]

This latter inequality holds for every \( x \) and \( y \). Hence the original inequality is true. Also we have equality at each step in the above argument, if and only if \( x = y \).

(b) (i) We will show by induction on \( n \) that

\[
a_n < a_{n+1} < b_{n+1} < b_n,
\]

for every \( n \).

This is true if \( n = 1 \), because then we have

\[
a_1 < a_2 = \sqrt{a_1 b_1} < b_2 = \frac{a_1 + b_1}{2} < b_1,
\]
by part (i). (Notice that \( a_1 < \sqrt{a_1 b_1} \) since \( a_1 < b_1 \).) Suppose now that
\[
a_k < a_{k+1} < b_{k+1} < b_k,
\]
Then
\[
a_{k+1} < a_{k+2} = \sqrt{a_k b_k}
\]
since \( a_k < b_k \).
Also \( a_{k+2} < b_{k+2} \), by Part (i).
Finally \( b_{k+2} < b_{k+1} \), since this latter number is the average of \( a_{k+1} \) and \( b_{k+1} > a_{k+1} \).
Hence the result holds for every \( n \) by induction.
Thus \( \{a_n\} \) is a monotonic increasing sequence which is bounded above . . . Every term is less than \( b_1 \). Also \( \{b_n\} \) is a monotonic decreasing sequence which is bounded below . . . Every term is greater than \( a_1 \). Hence both sequences have limits.
Suppose that \( a_n \to l_1 \) and \( b_n \to l_2 > l_1 \).
Then given \( \epsilon \), there exists \( N \) such that \( |b_n - l_1| < \frac{\epsilon}{2} \) and \( |a_n - l_1| \frac{\epsilon}{2} \), for every \( n > N \).
But then
\[
|b_{n+1} - \frac{l_1 + l_2}{2}| = |\frac{b_n + a_n}{2} - \frac{l_1 + l_2}{2}|
\leq \frac{1}{2}(|b_n - l_2| + |a_n - l_1|)
\leq \frac{\epsilon}{2} + \frac{\epsilon}{2}
< \epsilon.
\]
But this is impossible because then
\[
|b_{n+1} - l_2| > \frac{l_2 - l_1}{2} - \epsilon
\]
and this is obviously greater than \( \epsilon \), if we choose \( \epsilon \) small enough.
Hence \( l_1 = l_2 \).