1. For which of the following functions does the limit as $x \to 2$ exist?

(a) $f_1(x) = \begin{cases} 
1 & x \neq 2 \\
1.00001 & x = 2 
\end{cases}$

(b) $f_2(x) = \begin{cases} 
1 & x < 2 \\
1.00001 & x \geq 2 
\end{cases}$

(c) $f_3(x) = \begin{cases} 
1 & x < 2 \\
3 - x & x \geq 2 
\end{cases}$

(d) $f_4(x) = \begin{cases} 
1 & x < 2 \\
3 + x & x \geq 2 
\end{cases}$

**Solution**

(a) $\lim_{x \to 0} f_1(x) = 1$.

(b) $\lim_{x \to 0} f_2(x)$ does not exist.

As $x \to 2^-$, $f_2(x) \to 1$, while as $x \to 2^+$, $f_2(x) \to 1.00001$.

(c) $\lim_{x \to 0} f_3(x) = 1$.

(d) $\lim_{x \to 0} f_4(x)$ does not exist. As $x \to 2^-$, $f_4(x) \to 1$, while as $x \to 2^+$, $f_4(x) \to 5$.

2. If $f(x) = \begin{cases} 
4x^2 - 1 & x \neq -\frac{1}{2} \\
2x + 1 & \text{for } x = -\frac{1}{2}, \\
0 & \text{for } x = -\frac{1}{2} 
\end{cases}$ find $\lim_{x \to -\frac{1}{2}} f(x)$ and decide at which points $f$ is continuous.

**Solution**

If $x \neq -\frac{1}{2}$,

$$\frac{4x^2 - 1}{2x + 1} = \frac{(2x - 1)(2x + 1)}{2x + 1} = 2x - 1 \to -2, \text{ as } x \to -\frac{1}{2}.$$ 

Hence the function $f$ is not continuous at $x = -\frac{1}{2}$. It is continuous at all other points $x \neq -\frac{1}{2}$ as the quotient of two continuous functions in which the denominator is non-zero.

3. For each of the following functions, determine whether:

(a) $\lim_{x \to 0} f(x)$ exists  
(b) $f(x)$ is continuous at $x = 0$.

(c) $f(x)$ is differentiable at $x = 0$. 

(a) \( f_1(x) = \begin{cases} x, & \text{if } x \geq 0, \\ x^2, & \text{if } x < 0. \end{cases} \)

(b) \( f_2(x) = \begin{cases} x \cos \frac{1}{x}, & \text{if } x \neq 0, \\ 0, & \text{if } x = 0. \end{cases} \)

(c) \( f_3(x) = \begin{cases} x^2, & \text{if } x \geq 0, \\ x^3, & \text{if } x < 0. \end{cases} \)

(d) \( f_4(x) = \begin{cases} \frac{x - |x|}{x}, & \text{if } x \neq 0, \\ 0, & \text{if } x = 0. \end{cases} \)

**Solution**

(a) \( \lim_{x \to 0} f_1(x) = 0, \) because as \( x \to 0^- \), \( x \to 0 \), while as \( x \to 0^+ \), \( x^2 \to 0 \).

Since \( \lim_{x \to 0} f_1(x) = 0 = f(0) \), \( f \) is continuous at \( x = 0 \).

\( f_1 \) is not differentiable at \( 0 \) because the slope of the curve as \( x \to 0^- \) is 1, while for \( x \geq 0 \), the curve has slope \( 2x \) which approaches 0 as \( x \to 0^+ \).

(b) First

\[
0 \leq |x \cos \frac{1}{x}| \leq |x|, \text{ for every } x \neq 0,
\]

and by the Squeeze Law, \( x \cos \frac{1}{x} \to 0 \) as \( x \to 0 \). Hence \( \lim_{x \to 0} f_2(x) = 0 \).

Since \( f_2(0) = 0 \), \( f_2 \) is continuous at \( x = 0 \).

\[
\frac{f_2(x) - f_2(0)}{x - 0} = \frac{x \cos \frac{1}{x} - 0}{x - 0} = \cos \frac{1}{x},
\]

This last function, \( \cos \frac{1}{x} \) has no limit as \( x \to 0 \), since \( \cos \frac{1}{x} \) oscillates between 1 and -1 infinitely often in any neighbourhood of the origin.

Hence \( f_2 \) is not differentiable at the origin.

(c) \( \lim_{x \to 0} f_3(x) = 0 \) since as \( x \to 0^- \), \( x^3 \to 0 \), while as \( x \to 0^+ \), \( x^2 \to 0 \).

Hence \( f_3 \) is continuous at \( x = 0 \), because \( \lim_{x \to 0} f_3(x) = 0 = f_3(0) \).

\[
\frac{f_3(x) - f_3(0)}{x - 0} = \frac{x^2 - 0}{x - 0} \text{ for } x \geq 0
\]

\[
= x
\]

\[
\to 0, \text{ as } x \to 0^+,
\]

while

\[
\frac{f_3(x) - f_3(0)}{x - 0} = \frac{x^3 - 0}{x - 0} \text{ for } x < 0
\]

\[
= x^2
\]

\[
\to 0, \text{ as } x \to 0^-.
\]

Hence \( f_3 \) is differentiable at 0 and \( f_3'(0) = 0 \).

(d) If \( x > 0 \), \(|x| = x \) and so \( f_4(x) = 0 \). Hence as \( x \to 0^+ \), \( f_4(x) \to 0 \).

If \( x < 0 \), \(|x| = -x \), and so

\[
f_4(x) = \frac{x - |x|}{x} = \frac{2x}{x} = 2
\]

for \( x < 0 \). Hence as \( x \to 0^- \), \( f(x) \to 2 \).

Therefore \( \lim_{x \to 0} f_4(x) \) does not exist. and so \( f_4 \) is neither continuous nor differentiable at \( x = 0 \).
4. Give an example of a function \( f(x) \) defined and continuous on an interval \([a, b]\), with \( f(a) = f(b) = 0 \), but for which there is no point \( c \in (a, b) \) at which \( f'(c) = 0 \).

**Solution**

A function \( f \) which satisfies that conditions of the exercise and which is differentiable on the interval \([a, b]\), has a point \( c \) at which \( f'(c) = 0 \) by Rolle’s Theorem.

Hence to do this exercise, we need a function which is continuous on \([a, b]\) but not differentiable at every point of it.

The easiest example is the function \( f(x) = |x| - 1 \) on the interval \([-1, 1]\). Then \( f(-1) = f(1) = 0 \), but there is no point \( c \) in \([-1, 1]\), or anywhere else as it happens, at which \( f'(c) = 0 \). (Draw a graph!)

The function is not differentiable at \( x = 0 \).

5. Consider the function \( f(x) = x^3 - 3x^2 + 1 \).

(a) Verify Rolle’s Theorem on the interval \([0, 3]\).

(b) Verify the Mean Value Theorem on the interval \([0, 4]\).

(c) Notice that there are two points in \( \mathbb{R} \) at which the slope of the curve \( x^3 - 3x^2 + 1 \) equals \( \frac{f(4) - f(0)}{4} \) but only one of which lies in the interval \([0, 4]\).

**Solution**

(a) Note that \( f(0) = f(3) = 1 \). Hence there is a point \( c \) with \( 0 < c < 3 \) such that \( f'(c) = 0 \), by Rolle’s Theorem.

But \( f'(x) = 3x^2 - 6x = 3x(x - 2) \) and there are two points \( c \); namely 0, 2 at which \( f'(c) = 0 \). Only one of these points \( c = 2 \), lies in the interval \((0, 3)\).

(b) First \( f(0) = 1 \) and \( f(4) = 64 - 48 + 1 = 17 \).

The Mean Value Theorem asserts that there is a point \( c \in (0, 4) \) such that

\[
f'(c) = \frac{f(4) - f(0)}{4 - 0} = \frac{17 - 1}{4} = 4.
\]

Now \( f'(x) = 3x^2 - 6x \) and this equals 4 when \( 3x^2 - 6x - 4 = 0 \), that is when

\[
x = \frac{3 \pm \sqrt{21}}{3} = 2.52 \ldots \text{ or } -0.52 \ldots.
\]

The required point is \( c = \frac{3 + \sqrt{21}}{3} \).

(c) The other point \( \frac{3 - \sqrt{21}}{3} = -0.52 \ldots \) does not lie in the interval \([0, 4]\).

6. An aircraft takes off from Sydney at 11:00 am. At 1.00pm Eastern Standard Time (EST), it is 1900 km from Sydney and at 3.30 pm (EST), it is 4260 km from Sydney. Show that at some time between 1.00pm and 3.30 pm its speed was 944 kph.

**Solution**

We can assume that the distance of the plane from Sydney as a function \( f(t) \) of time is differentiable as many times as we like. Hence the Mean Value Theorem applies to it.

Hence there exists a time \( c \) between 1.00pm and 3.30pm such that

\[
\frac{f(3.30) - f(1.00)}{2.5} = f'(c).
\]
Hence
\[ f'(c) = \frac{4260 - 1900}{2.5} = 944. \]

Thus at some time between 1.00 pm and 3.30 pm EST, the plane was travelling at 944 kph.

7. Suppose that \( f(x) = \begin{cases} 0 & \text{if } x \text{ is irrational} \\ \frac{1}{q} & \text{if } x = \frac{p}{q} \text{ is rational with } \frac{p}{q} \text{ in reduced form} \end{cases} \)

(a) Show that \( f \) is discontinuous at all rational points.

(b) (Harder) In fact \( f \) is continuous at every irrational point.

**Solution**

(a) Given any rational number \( x = \frac{p}{q} \), there are infinitely many rational numbers as close to it as we please. These rational numbers will have the form \( \frac{r}{s} \) in reduced form and it will be numbers with denominators \( s \) large which will be closest to \( x \). For example, if \( x = \frac{1}{2} \), the following sequence \( \{a_n\} \) has \( x \) as its limit.

\[
\begin{align*}
49 & \quad 499 \\
50 & \quad 500 \\
\end{align*}
\]

Note that these numbers are all in reduced form.

Notice that \( \{a_n\} \to \frac{1}{2} \) as \( n \to \infty \) but that

\[
f\left(\frac{1}{2}\right) = \frac{1}{2} \\
\neq \lim_{n \to \infty} f(a_n) \\
= \lim_{n \to \infty} \frac{1}{5 \times 10^n} \\
= 0.
\]

Hence \( f \) is not continuous at any rational point.

(b) Amazingly, \( f \) is continuous at irrational points!

Suppose that \( a \) is any irrational number. Given any number \( \epsilon > 0 \), we want to show that \( \lim_{x \to a} f(x) = f(a) = 0 \). That is we want to show that we can find a number \( \delta \) such that

\[ |f(x) - f(a)| = |f(x) - 0| = |f(x)| < \epsilon \text{ when } |x - a| < \delta. \]

Now \( f(x) = 0 \) for any irrational number \( x \) and so the only way that \( |f(x) - f(a)| \) is not less than \( \epsilon \) is if \( x \) is rational.

Which rational numbers \( x = \frac{p}{q} \) have the property that \( |f(x) - f(a)| = |f\left(\frac{p}{q}\right)| = \frac{1}{|q|} > \epsilon? \)

These are numbers for which \( |q| < \frac{1}{\epsilon} \).

There are only finitely many such numbers \( q \).

The rational numbers \( \frac{p}{q} \) (in reduced form) with denominators \( q \) with \( |q| < \frac{1}{\epsilon} \) have a minimum non-zero distance apart and so are separated from \( a \) by some positive (maybe small!) non-zero distance \( d \).
Choose $\delta < d$. Then if $|x - a| < \delta$, either $x$ is irrational and $f(x) = 0$ or $x$ is rational, in which case its denominator (in reduced form) is greater than $|q|$. But then

$$|f(x) - f(a)| = |f\left(\frac{p}{q}\right)| = \frac{1}{q} < \epsilon.$$ 

Hence $|f(x) - f(a)| < \epsilon$ for all $x$ with $|x - a| < \delta$ and so $f$ is continuous at every irrational point $x$. 