1. (a) Find the radius of convergence \( R \) of the power series \( \sum_{k=1}^{\infty} \frac{x^{3k}}{k} \).

(b) Let \( f(x) \) be the sum of the series in (a) for \( |x| < R \). Calculate \( f'(x) \).

(c) Use part (b) to find \( f(x) \).

**Solution**

(a) Use the ratio test.

\[
\left| \frac{a_{n+1}}{a_n} \right| = \left| \frac{x^{3(k+1)}}{k+1} \frac{k}{x^{3k}} \right| = \frac{k+1}{k} |x|^3
\]

\( \rightarrow |x|^3 < 1, \text{ as } x \to \infty. \)

Hence the radius of convergence is 1.

(b) First if \( |x| < 1 \),

\[ f(x) = \frac{x^3}{1} + \frac{x^6}{2} + \frac{x^9}{3} + \frac{x^{12}}{4} + \ldots \]

and so,

\[ f'(x) = \frac{3x^2}{1} + \frac{6x^5}{2} + \frac{9x^8}{3} + \frac{12x^{11}}{4} + \ldots \]

\[ = 3x^2 + 3x^5 + 3x^8 + \ldots \]

\[ = 3x^2(1 + x^3 + x^6 + \ldots), \text{ if } |x| < 1. \]

(c) Since \( 1 + x^3 + x^6 + x^9 + \ldots = \frac{1}{1-x^3} \), if \( |x| < 1 \), it follows that

\[ f'(x) = \frac{3x^2}{1-x^3} \]

and so \( f(x) \) and

\[
\int_0^x \frac{3t^2}{1-t^3} \, dt = - \int_1^{1-x^3} \frac{du}{u}
\]

\[ = - \ln u \bigg|_{1}^{1-x^3} \]

\[ = - \ln(1-x^3). \]

Since when \( x = 0, f(x) = 0 \) and \( \ln(1-x^3) = 0 \), it follows that

\[ f(x) = - \ln(1-x^3) \]

.
2. Find two linearly independent power series solutions \( \sum a_n x^n \) of the differential equations. Calculate also their radii of convergence.
(a) \( y'' - y = 0 \),
(b) \( y'' - x^2 y = 0 \),
(c) \( y'' - 2xy' + y = 0 \). (This is called Hermite's equation and arises in quantum mechanics in the solution of the Schrödinger equation for a harmonic oscillator.)

**Solution**

(a)

\[
y = a_0 + a_1 x + a_2 x^2 + \ldots + a_n x^n + \ldots
\]
\[
y' = a_1 + 2a_2 x + 3a_3 x^2 + \ldots + (n+1)a_{n+1} x^n + \ldots
\]
\[
y'' = 2a_2 + 3.2a_3 x + 4.3a_4 x^2 + \ldots + (n+2)(n+1)a_{n+2} x^n + \ldots
\]

Hence

\[
y'' = 2a_2 + 3.2a_3 x + 4.3a_4 x^2 + \ldots + (n+2)(n+1)a_{n+2} x^n + \ldots
\]

\[-y = -a_0 - a_1 x - a_2 x^2 - \ldots - a_n x^n - \ldots
\]

Adding we get

\[
2.1a_2 - a_0 = 0
\]
\[
3.2a_3 - a_1 = 0
\]
\[
4.3a_4 - a_2 = 0
\]
\[
\vdots
\]
\[
(n+2)(n+1)a_{n+2} - a_n = 0.
\]

Hence

\[
a_2 = \frac{1}{2!}a_0, \quad a_4 = \frac{1}{4.3}a_2 = \frac{1}{4!}a_0, \ldots, a_{2n} = \frac{1}{2n!}a_0
\]
\[
a_3 = \frac{1}{3!}a_1, \quad a_5 = \frac{1}{5.4}a_3 = \frac{1}{5!}a_1, \ldots, a_{2n+1} = \frac{1}{(2n+1)!}a_1.
\]

Hence

\[
y = a_0(1 + \frac{x^2}{2!} + \frac{x^4}{4!} + \ldots) +
\]
\[
a_1(x + \frac{x^3}{3!} + \frac{x^5}{5!} + \ldots)
\]

Both series converge for all \( x \) by the ratio test. Remark.

You have learned in First Year that the solution of the differential equation can be written in the form \( ae^x + be^{-x} \) since the characteristic equation of the differential equation is \( \lambda^2 - 1 \) which has eigen-values \( \pm 1 \). This leads to the eigen-functions \( e^x \) and \( e^{-x} \).

We get our series solution from this as follows:

\[
ae^x + be^{-x} = a(1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \ldots) +
\]
\[
b(1 - x + \frac{x^2}{2!} + \frac{x^3}{3!} - \ldots) = (a + b)(1 + \frac{x^2}{2!} + \frac{x^4}{4!} + \ldots) +
\]
\[
(a - b)(x + \frac{x^3}{3!} + \frac{x^5}{5!} + \ldots).
\]
(b) 

\[ y = a_0 + a_1 x + a_2 x^2 + \ldots + a_n x^n + \ldots \]
\[ y' = a_1 + 2a_2 x + 3a_3 x^2 + \ldots + (n+1)a_{n+1} x^n + \ldots \]
\[ y'' = 2a_2 + 3.2a_3 x + 4.3a_4 x^2 + \ldots + (n+2)(n+1)a_{n+2} x^n + \ldots \]

Hence

\[ y'' = 2a_2 + 3.2a_3 x + 4.3a_4 x^2 + \ldots + (n+2)(n+1)a_{n+2} x^n + \ldots \]
\[ -x^2 y = -a_0 x^2 - a_1 x^3 - a_2 x^4 + \ldots - a_{n-2} x^n - \ldots \]

Hence

\[ a_2 = 0 \]
\[ a_3 = 0 \]
\[ 4.3a_4 - a_0 = 0 \]
\[ 5.4a_5 - a_1 = 0 \]
\[ 6.5a_6 - a_2 = 0 \]
\[ 7.6a_7 - a_3 = 0 \]

\[ \vdots \]
\[ (n+2)(n+1)a_{n+2} - a_n = 0. \]

Hence

\[ a_2 = a_3 = a_6 = a_7 = \ldots = a_{4n+2} = a_{4n+3} = 0 \]

and

\[ a_4 = \frac{1}{4.3}a_0, \quad a_8 = \frac{1}{8.7}a_4 = \frac{1}{8.7.4.3}a_0, \quad \ldots \]

and

\[ a_5 = \frac{1}{5.4}a_1, \quad a_9 = \frac{1}{9.8}a_5 = \frac{1}{9.8.5.4}a_1, \quad \ldots \]

Hence

\[ y = a_0 (1 + \frac{1}{4.3} x^3 + \frac{1}{8.7.4.3} x^8 + \ldots) + \]
\[ a_1 (x + \frac{1}{5.4} x^5 + \frac{1}{9.8.5.4} x^9 + \ldots) \]

Both series converge for all \( x \) by the ratio test.

(c) 

\[ y = a_0 + a_1 x + a_2 x^2 + \ldots + a_n x^n + \ldots \]
\[ y' = a_1 + 2a_2 x + 3a_3 x^2 + \ldots + (n+1)a_{n+1} x^n + \ldots \]
\[ y'' = 2a_2 + 3.2a_3 x + 4.3a_4 x^2 + \ldots + (n+2)(n+1)a_{n+2} x^n + \ldots \]

Hence

\[ y'' = 2a_2 + 3.2a_3 x + 4.3a_4 x^2 + \ldots + (n+2)(n+1)a_{n+2} x^n + \ldots \]
\[ -2xy' = -2a_1 x - 2.2a_2 x^2 - 2.3a_3 x^3 - \ldots - 2na_n x^n - \ldots \]
\[ y = a_0 + a_1 x + a_2 x^2 + \ldots + a_n x^n + \ldots \]

Hence we have

\[ 2.1a_2 + a_0 = 0 \]
\[ 3.2a_3 - a_1 = 0 \]
\[ (n+2)(n+1)a_{n+2} - (2n-1)a_n = 0. \]
Therefore
\begin{align*}
a_2 &= \frac{-1}{2!}a_0 \\
a_4 &= \frac{3}{4!}a_2 = \frac{3}{4!}a_0 \\
a_6 &= \frac{7}{6!}a_4 = \frac{7}{6!}a_0 \\
a_8 &= \frac{7}{6!}a_4 = \frac{11.7.3}{8!}a_0 \\
\vdots \\
a_3 &= \frac{1}{3!}a_1 \\
a_5 &= \frac{5}{5!}a_3 = \frac{5}{5!}a_1 \\
a_7 &= \frac{9}{7!}a_5 = \frac{9.5.1}{7!}a_1 \\
a_9 &= \frac{13.9.5.1}{9!}a_1 \\
\vdots
\end{align*}

Hence the solution is
\begin{align*}
y &= a_0 \left( 1 - \frac{1}{2}x^2 - \frac{3}{4!}x^4 - \frac{7.3}{6!}x^6 - \ldots \right) + \\
&\quad \quad a_1 \left( x + \frac{1}{3!}x^3 + \frac{5}{5!}x^5 + \frac{9.5.1}{7!}x^7 + \ldots \right).
\end{align*}

Both these series have infinite radii of convergence by the ratio test. For example, in the first series
\begin{align*}
\frac{a_{2n+2}}{a_{2n}} &= \left| \frac{x^{2n+2} \frac{3.7 \ldots (4n-5)}{(2n+2)!} \frac{x^{2n}(2n)!}{3.7 \ldots (4n-9)}}{(2n+2)! \frac{4n-5}{(2n+2)(2n+1)}} \right| \\
&= \left| x^2 \frac{4n-5}{(2n+2)(2n+1)} \right| \\
&\to 0, \text{ as } n \to \infty.
\end{align*}

3. Find the first three terms in each of two linearly independent power series solutions in powers of \( x \) of \( y'' + y \sin x = 0 \).

\textbf{Solution}
\begin{align*}
y &= a_0 + a_1 x + a_2 x^2 + \ldots + a_n x^n + \ldots \\
y' &= a_1 + 2a_2 x + 3a_3 x^2 + \ldots + (n+1)a_{n+1} x^n + \ldots \\
y'' &= 2.1a_2 + 3.2a_3 x + 4.3a_4 x^2 + \ldots + (n+2)(n+1)a_{n+2} x^n + \ldots.
\end{align*}

Hence considering
\begin{equation*}
y'' + \left( x - \frac{x^3}{3!} + \frac{x^5}{5!} - \ldots \right)y
\end{equation*}
we have
\[ 0 = 2.1a_2 + 3.2a_3x + 4.3a_4x^2 + 5.4a_5x^3 + 6.5a_6x^4 + 7.6a_7x^5 + 8.6a_8x^6 + \]
\[ + a_0x + a_1x^2 + a_2x^3 + a_3x^4 + a_4x^5 + a_5x^6 + \]
\[ - \frac{1}{3!}a_0x^3 - \frac{1}{3!}a_1x^4 - \frac{1}{3!}a_2x^5 - \frac{1}{3!}a_3x^6 - \]
\[ + \frac{1}{5!}a_0x^5 + \frac{1}{5!}a_1x^6. \]

\[ \ldots \]

Hence
\[ a_2 = 0 \]
\[ 6a_3 + a_0 = 0 \]
\[ 12a_4 + a_1 = 0 \]
\[ 20a_5 + a_2 - \frac{1}{3!}a_0 = 0 \]
\[ 30a_6 + a_3 - \frac{1}{3!}a_1 = 0 \]
\[ 42a_7 + a_4 - \frac{1}{6}a_2 + \frac{1}{120}a_0 = 0. \]

Hence
\[ a_2 = 0 \]
\[ a_3 = -\frac{1}{6}a_0 \]
\[ a_4 = -\frac{1}{12}a_1 \]
\[ a_5 = \frac{1}{120}a_0 \]
\[ a_6 = \frac{1}{180}a_1 + \frac{1}{180}a_0. \]

Hence
\[ y = a_0 + a_1x - \frac{1}{6}a_0x^3 - \frac{1}{12}a_1x^4 + \frac{1}{120}a_0x^5 + \left( \frac{1}{180}a_1 + \frac{1}{180}a_0 \right)x^6 + \ldots \]
\[ = a_0(1 - \frac{1}{6}x^3 + \frac{1}{120}x^5 - \frac{1}{180}x^6 + \ldots) \]
\[ + a_1(x - \frac{1}{12}x^4 + \frac{1}{180}x^6 + \ldots). \]

4. Classify each of the following functions as odd, even or neither:

(a) \( f(x) = x^3 + 2x \)
(b) \( f(x) = \sin x + \cos x \)
(c) \( f(x) = x \sin x \)
(d) \( f(x) = x^2 \sin x \)
(e) \( f(x) = |x + 2| \)
(f) \( f(x) = \frac{|x|}{x} \)

**Solution**

(a) \( f(-x) = (-x)^3 + 2(-x) = -(x^3 + 2x) = -f(x) \) and so \( f \) is odd.
(b) \( f(-x) = \sin(-x) + \cos(-x) = -\sin x + \cos x \). This is not of the form \( f(x) \) for all \( x \) nor is it of the form \(-f(x)\) and so \( f \) is neither even nor odd.
(c) \( f(-x) = (-x)\sin(-x) = x \sin x = f(x) \) and so \( f \) is even.
(d) \( f(-x) = (-x)^2 \sin(-x) = -x^2 \sin x = -f(x) \) and so \( f \) is odd.

(e) \( f(-x) = | -x + 2| \) which is neither of the form \(|x + 2|\) nor of the form \(-|x + 2|\). Hence \( f \) is neither even nor odd.

(f) \( f(-x) = \frac{|-x|}{-x} = -\frac{|x|}{x} = -f(x) \) and so \( f \) is odd. (I am assuming that we define \( f(0) \) to be 0 here, otherwise \( f \) is not odd.

5. What is the smallest positive period of the following functions?
   \( (a) \ \cos 3x \quad (b) \ \sin \frac{1}{5}x \quad (c) \ \tan x \quad (d) \ \cos^2 2x + \sin 4x. \)

\textit{Solution}

(a) \( \cos 3x \) has period \( \frac{2\pi}{3} \) since
\[
\cos 3(x + \frac{2\pi}{3}) = \cos(3x + 2\pi) = \cos 3x.
\]
This is also the smallest period.

(b) \( \sin \frac{1}{5}x \) has period \( 10\pi \) since
\[
\sin \frac{1}{5}(x + 10\pi) = \sin(\frac{1}{5}x + 2\pi) = \sin \frac{1}{5}x.
\]

(c) \( \tan x \) has smallest period \( \pi \).

(d)
\[
\cos^2 2x + \sin 4x = \frac{1 + \cos 4x}{2} + \sin 4x.
\]
Hence the smallest period is \( \frac{2\pi}{4} = \frac{\pi}{2} \).

6. Find the Fourier series expansions of the \( 2\pi \)-periodic functions
   \( (a) \ \sin^2 x + \cos 2x \quad (b) \ \cos^2 \frac{1}{2}x \quad (c) \ \sin^3 x. \)

\textit{Solution}

(a) The Fourier series is given by:
\[
\sin^2 x + \cos 2x = \frac{1 - \cos 2x}{2} + \cos 2x
= \frac{1}{2} + \frac{1}{2} \cos 2x.
\]

(b)
\[
\cos^2 \frac{1}{2}x = \frac{1 + \cos x}{2}
= \frac{1}{2} + \frac{1}{2} \cos x.
\]

(c) First \( \sin 3x = 3 \sin x - 4 \sin^3 x \) and so
\[
\sin^3 x = \frac{1}{4} (3 \sin x - \sin 3x) = \frac{3}{4} \sin x - \frac{1}{4} \sin 3x.
\]