Assignment 4

“Regular surfaces”

Due Thursday, 22 September, at 16:00 in the assignment box for MATH3405. The box is located on Level 4 in the Mathematics (Priestley) building (67). It is number 035 in the brown lot of boxes (there are two lots, to find ours, turn right as you come from the stairs).

Please use a cover sheet!

Clearly state your assumptions and conclusions, and justify all steps in your work. Marks will be deducted for sloppy or incomplete working.

Q1 Let $U = \{(u,v) \in \mathbb{R}^2 \mid u^2 + v^2 < 1\}$, and $S^2 = \{(x,y,z) \in \mathbb{R}^3 \mid x^2 + y^2 + z^2 = 1\}$.
Let $\Phi: U \to S^2$ and $\Psi: U \to S^2$ be the local coordinate charts defined by:

$$\Phi(u,v) = (u,v,\sqrt{1-u^2-v^2}), \quad \Psi(u,v) = (v,\sqrt{1-u^2-v^2},u).$$

(a) Determine the subsets $\Phi(U)$, $\Psi(U)$ and $\Phi(U) \cap \Psi(U)$ of $S^2$.
(b) Determine the domain of $\Psi^{-1} \circ \Phi$ and find a closed expression for this function.
(c) Determine the domain of $\Phi^{-1} \circ \Psi$ and find a closed expression for this function.

Q2 Let $\Phi: U \to \mathbb{R}^3$ be a parametrised surface, where $U \subseteq \mathbb{R}^2$ is open. Since $\Phi_u = \frac{\partial \Phi}{\partial u}$ and $\Phi_v = \frac{\partial \Phi}{\partial v}$ are well-defined at each point in $U$, we can define the real-valued functions

$$E = E(u,v) = \Phi_u \cdot \Phi_u, \quad F = F(u,v) = \Phi_u \cdot \Phi_v, \quad G = G(u,v) = \Phi_v \cdot \Phi_v,$$

where $\cdot$ is the dot product in $\mathbb{R}^3$.

Show that $\Phi$ is regular if and only if the function $EG - F^2$ is nowhere zero.

Q3 Let $S_1$ and $S_2$ be regular surfaces. Suppose that $S_1$ has a local coordinate chart $\Psi: U \to S_1$ with the property that $S_1 = \text{Im}(\Psi)$.

Given any differentiable map $f: U \to S_2$, show that the map $F: S_1 \to S_2$ defined by

$$F(\Psi(u,v)) = f(u,v) \quad \text{for all } (u,v) \in U$$

is a differentiable map of surfaces.

Q4 Let $R > r > 0$ and define $\Phi: \mathbb{R}^2 \to \mathbb{R}^3$ by

$$\Phi(\theta,\varphi) = ((R + r \cos \theta) \cos \varphi, (R + r \cos \theta) \sin \varphi, r \sin \theta).$$

Let $S = \text{Im}(\Phi)$. You know that this is a torus, and you may assume that it is a smooth regular surface. Since the torus is compact, it cannot be covered by a single local coordinate chart.

(a) Determine two or more local coordinate charts whose union covers the torus.
(b) Give a basis for the tangent space $T_pS$ for each $p \in S$.
(c) Compute the first fundamental form of $S$ at each $p \in S$. 
[Mapping the helicoid to the torus]

An application of Q3 (not part of the assignment): Let $b \neq 0$ and define $\Psi : \mathbb{R}^2 \to \mathbb{R}^3$ by

$$\Psi(u, v) = (u \cos v, u \sin v, bv).$$

Then $\Psi$ is injective and $H = \text{Im}(\Psi)$ is a regular surface called the helicoid (a picture of it is on the course web-site). It can be viewed as the disjoint union of lines $L_v$ parallel to the xy-plane, obtained by fixing $v$ and varying $u$. These are called rulings.

Let $R > r > 0$ and define $\Phi : \mathbb{R}^2 \to \mathbb{R}^3$ by

$$\Phi(\theta, \phi) = ((R + r \cos \theta) \cos \phi, (R + r \cos \theta) \sin \phi, r \sin \theta).$$

Then $\Psi$ is not injective and its image $T = \text{Im}(\Phi)$ is a torus. This can be viewed as the disjoint union of circles $C_\theta$, obtained by fixing $\phi$ and varying $\theta$. These are called meridian circles.

Define a differentiable map $H \to T$ such that rulings are mapped to meridian circles.