Problem Set 4

Q27 Let $X$ be a normed space and $x_0 \in X$. Show that $\|x_0\| \leq C$ if and only if $|f(x_0)| \leq C$ for all $f \in S(X^*)$.

Q28 Complete P2 Q15, showing that $\|T\| = \|T^*\|$, i.e. the map $T \to T^*$ is an isometry, using one of the consequences of the Hahn-Banach theorem.

Q29 Show that if $X$ is reflexive, then $X^*$ is reflexive. What about the converse?

Q30 Prove the following facts about the sequence spaces $l^p$.
   (a) For $1 < p < \infty$, $(l_p)^*$ is isometric with $l_q$, where $q$ satisfies $\frac{1}{p} + \frac{1}{q} = 1$.
      Conclude that $l_p$ is reflexive.
   (b) The dual $(c_0)^*$ is isometric with $l_1$.
   (c) The dual $(l_1)^*$ is isometric with $l_\infty$.
   (d) The spaces $c_0$, $l_1$ and $l_\infty$ are not reflexive.

Q31 Let $X$ be a normed space.
   (a) What are $X^\perp$ and $\{0\}^\perp$?
   (b) If $Y_1, Y_2$ are closed subspaces of $X$ such that $Y_1 \neq Y_2$, show that $Y_1^\perp \neq Y_2^\perp$.
      Is this also true if one or both subspaces are not closed?

Q32 Let $Y$ be a subspace of the normed space $X$. Suppose $\dim X = n$ and $\dim Y = m$.
Show that $\dim Y^\perp = n - m$.
Formulate this as a theorem about the solution set of a system of linear equations.

Q33 Let $T \in \mathfrak{B}(X,Y)$. Show that:
   (a) $(\text{im}(T))^\perp \subseteq \ker(T^*)$
   (b) $\text{im}(T) \subseteq \ker(T^*)^\perp$.
What does the second part imply for solving $Tx = y$?

Q34 Let $X$ be a set and $\mathcal{F}_b(X)$ be the real vector space of all bounded, real-valued functions on $X$. For each $f \in \mathcal{F}_b(X)$, let $||f|| = \sup_{x \in X} |f(x)|$.
You may assume that $(\mathcal{F}_b(X), || \cdot ||)$ is a normed space.
   (a) Show that $(\mathcal{F}_b(X), || \cdot ||)$ is complete.
   (b) Suppose that $(X, d)$ is a metric space and fix $a \in X$. For $x \in X$, define the function $f_x: X \to \mathbb{R}$ by:
      $$f_x(t) = d(x, t) - d(a, t).$$
      Show that $f_x \in \mathcal{F}_b(X)$, and that the map $x \to f_x$ is an isometry onto its image (where $\mathcal{F}_b(X)$ is given the metric induced by the norm). Conclude that $X$ is homeomorphic to a subset of $\mathcal{F}_b(X)$, where both spaces are given the induced topologies.
   (c) Discuss differences and analogies of (b) with the result that every normed space is isometric to a subspace of its double-dual.