Problem Set 5

Q35 Show that a compact subset of a normed space is closed and bounded.

Q36 Let $Y$ be a closed, proper subspace of the normed space $X$.
Show that for every $\varepsilon > 0$, there is a point $x \in S(X)$ such that $\text{dist}(x, Y) \geq 1 - \varepsilon$.

Q37 Show that the (closed) unit ball in $l^n_1$ is compact.

Q38 Deduce from Lemma 4.8 that a Banach space cannot have a countably infinite algebraic basis;
i.e. if $X = \text{span}\{b_k|k \in \mathbb{N}\}$ and $\{b_k|k \in \mathbb{N}\}$ is linearly independent, then $X$ is incomplete.
(The notation $[\ldots]$ indicates that a basis is not a set, but rather a system of vectors.)

Q39 Find normed spaces that are
(a) algebraically isomorphic, but not topologically isomorphic;
(b) topologically isomorphic, but not isometrically isomorphic.

Q40 How is the Banach–Mazur distance between $X$ and $Y$ related to the Banach–Mazur distance between $X^*$ and $Y^*$?

Q41 Let $X$ and $Y$ be finite dimensional normed spaces over the same field and with the same dimension. Show that the Banach–Mazur distance between $X$ and $Y$, $d(X, Y)$, is attained.