ELSEVIER

Discrete Mathematics 152 (1996) 1–12

DISCRETE MATHEMATICS

A generalization of the Ahlswede–Daykin inequality

Ron Aharoni*, Uri Keich

Department of Mathematics, Technion-Israel Institute of Technology, 32000 Haifa, Israel

Received 16 July 1992; revised 14 December 1993

Abstract

We prove an inequality concerning two n-tuples of nonnegative functions on a distributive lattice, of which the Ahlswede–Daykin inequality is the case $n = 2$. This is a rediscovery: Rinott and Saks proved the same inequality a little earlier. But our approach and proofs are somewhat different.

1. Introduction

In 1977 Daykin [4] proved an inequality on distributive lattices, which has since proved to be very useful. To state it, let us introduce the following notation. Let $\mathscr{L}$ be a distributive lattice and let $\mathscr{A}, \mathscr{B} \subseteq \mathscr{L}$. Denote by $\mathscr{A} \vee \mathscr{B}$ the set $\{A \vee B \colon A \in \mathscr{A}, B \in \mathscr{B}\}$ and by $\mathscr{A} \wedge \mathscr{B}$ the set $\{A \wedge B \colon A \in \mathscr{A}, B \in \mathscr{B}\}$. Daykin's inequality then says that $|\mathscr{A} \vee \mathscr{B}|\,|\mathscr{A} \wedge \mathscr{B}| \geqslant |\mathscr{A}|\,|\mathscr{B}|$. Here is, for example, one application: Suppose that $\mathscr{A}$ and $\mathscr{B}$ are filters (= up-sets) in $\mathscr{L}$, i.e. if $A \in \mathscr{A}$ and $C \geqslant A$ then $C \in \mathscr{A}$, and similarly for $\mathscr{B}$. Then clearly $\mathscr{A} \vee \mathscr{B} = \mathscr{A} \cap \mathscr{B}$. Hence $|\mathscr{A} \cap \mathscr{B}| \geqslant |\mathscr{A}|\,|\mathscr{B}|/|\mathscr{A} \wedge \mathscr{B}| \geqslant |\mathscr{A}|\,|\mathscr{B}|/|\mathscr{L}|$, or

$$\frac{|\mathscr{A} \cap \mathscr{B}|}{|\mathscr{L}|} \geqslant \frac{|\mathscr{A}|}{|\mathscr{L}|}\frac{|\mathscr{B}|}{|\mathscr{L}|}.$$

This means that $\mathscr{A}$ and $\mathscr{B}$ are 'positively correlated' (in the uniform probability on $\mathscr{L}$ one has $\Pr(X \in \mathscr{A}) = |\mathscr{A}|/|\mathscr{L}|$). This is an extension of Kleitman's inequality [7], which states the same for the lattice of subsets of a given set. One can also derive from this result the FKG inequality [6], which states that two ascending functions on a distributive lattice are positively correlated.

Soon thereafter Ahlswede and Daykin [1] found an extension of this result. If α is a function on $\mathscr{L}$, and $\mathscr{A} \subseteq \mathscr{L}$, write $\alpha(\mathscr{A}) = \sum\{\alpha(A) \colon A \in \mathscr{A}\}$. Suppose that $\alpha^1, \alpha^2, \beta^1$

* Corresponding author.

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and β^2 are nonnegative functions on $\mathscr{L}$, satisfying

$$\alpha^1(A)\alpha^2(B) \leqslant \beta^1(A \vee B)\beta^2(A \wedge B) \tag{1.1}$$

for every pair $A, B \in \mathscr{L}$. The Ahlswede–Daykin inequality says then that for every pair $\mathscr{A}$, $\mathscr{B}$ of subsets of $\mathscr{L}$ one has

$$\alpha^1(\mathscr{A})\alpha^2(\mathscr{B}) \leqslant \beta^1(\mathscr{A} \vee \mathscr{B})\beta^2(\mathscr{A} \wedge \mathscr{B}). \tag{1.2}$$

Daykin's inequality is obtained by taking $\alpha^1 = \alpha^2 = \beta^1 = \beta^2 \equiv 1$.

One can wonder why just four functions appear in this inequality, and whether the inequality can be extended to two n-tuples of nonnegative functions. We shall present such an extension in this paper.

Let $A = (A_1, A_2, \ldots, A_n)$ be a vector of elements of the distributive lattice $\mathscr{L}$. For each $1 \leqslant k \leqslant n$ write $\binom{[n]}{k} = \{S \subset [n], |S| = k\}$ (here and henceforth $[n]$ denotes the set $\{1, \ldots, n\}$) and let

$$f_k(A) = \bigvee \left\{ \bigwedge_{i \in S} A_i \colon S \in \binom{[n]}{k} \right\}.$$

For example, if $A = (A, B, C)$ then $f_1(A) = A \vee B \vee C$, $f_2(A) = (A \wedge B) \vee (A \wedge C) \vee (B \wedge C)$ and $f_3(A) = A \wedge B \wedge C$.

Assume now that $\mathscr{A} = (\mathscr{A}_1, \mathscr{A}_2, \ldots, \mathscr{A}_n)$ is a vector of subsets $\mathscr{A}_i$ of $\mathscr{L}$. For a vector $A = (A_1, \ldots, A_n)$ of elements of $\mathscr{L}$ write $A \subsetneq \mathscr{A}$ if $A_i \in \mathscr{A}_i$ for all $1 \leqslant i \leqslant n$. Then let $f_k(\mathscr{A}) = \{f_k(A) \colon A \subsetneq \mathscr{A}\}$.

Theorem 1.1. *Let $\alpha^1, \alpha^2, \ldots, \alpha^n$ and $\beta^1, \beta^2, \ldots, \beta^n$ be nonnegative functions on a distributive lattice $\mathscr{L}$. Suppose that for every vector $A = (A_1, A_2, \ldots, A_n)$ of elements of $\mathscr{L}$ one has*

$$\prod_{1 \leqslant i \leqslant n} \alpha^i(A_i) \leqslant \prod_{1 \leqslant i \leqslant n} \beta^i(f_i(A)). \tag{1.3}$$

Then, for every vector $\mathscr{A} = (\mathscr{A}_1, \ldots, \mathscr{A}_n)$ of subsets of $\mathscr{L}$

$$\prod_{1 \leqslant i \leqslant n} \alpha^i(\mathscr{A}_i) \leqslant \prod_{1 \leqslant i \leqslant n} \beta^i(f_i(\mathscr{A})). \tag{1.4}$$

The proof of Theorem 1.1, given in Section 3, will use an inequality of a geometrical flavour (Theorem 2.1), proved in Section 2. In Section 4 we shall use, in turn, Theorem 1.1 to derive a generalization of Theorem 2.1.

As mentioned in the abstract, while writing a first version of this paper, two papers of Rinott and Saks [9, 10] were brought to our attention. They were written a few months earlier, and basically contain the same results (even down to the conjectures ...). Our terminology and basic approach are, however, so different from that of the Rinott–Saks papers, that we decided to publish nevertheless.

2. A geometric inequality

Let us consider the Ahlswede–Daykin inequality for the lattice $\mathscr{L} = \mathscr{P}([1]) = \{\emptyset, [1]\}$. Write $\alpha^1(0) = \alpha^1(\phi)$, $\alpha^1(1) = \alpha^1([1])$ and similarly for α^2, β^1 and β^2. Then (1.1) yields

$$\begin{aligned} &\alpha^1(0)\alpha^2(0) \leqslant \beta^1(0)\beta^2(0), \quad &&\alpha^1(0)\alpha^2(1) \leqslant \beta^1(1)\beta^2(0), \\ &\alpha^1(1)\alpha^2(0) \leqslant \beta^1(1)\beta^2(0), \quad &&\alpha^1(1)\alpha^2(1) \leqslant \beta^1(1)\beta^2(1). \end{aligned} \tag{2.1}$$

Take (1.2) and put $\mathscr{A} = \mathscr{B} = \mathscr{L}$:

$$(\alpha^1(0) + \alpha^1(1))\,(\alpha^2(0) + \alpha^2(1)) \leqslant (\beta^1(0) + \beta^1(1))\,(\beta^2(0) + \beta^2(1)). \tag{2.2}$$

The implication (2.1) $\Rightarrow$ (2.2) has a geometric interpretation. Let Q and R be two rectangles of areas q and r, respectively. Divide each into 4 subrectangles, as follows:

$$Q\colon \begin{array}{c|c|c|} & \alpha^2(0) & \alpha^2(1) \\ \hline \alpha^1(0) & q_{00} & q_{01} \\ \hline \alpha^1(1) & q_{10} & q_{11} \\ \hline \end{array} \qquad R\colon \begin{array}{c|c|c|} & \beta^2(0) & \beta^2(1) \\ \hline \beta^1(0) & r_{00} & r_{01} \\ \hline \beta^1(1) & r_{10} & r_{11} \\ \hline \end{array}$$

where the letters q_{ij}, r_{ij} refer to the areas of the corresponding subrectangles. The implication (2.1) $\Rightarrow$ (2.2) says that if $q_{00} \leqslant r_{00}$, $q_{01} \leqslant r_{10}$, $q_{10} \leqslant r_{10}$ and $q_{11} \leqslant r_{11}$ then $q \leqslant r$. This would be easy to prove if one replaced the inequality $q_{01} \leqslant r_{10}$ by $q_{01} \leqslant r_{01}$, but is also not difficult to prove as it is.

We would like to prove an extension of this inequality for any number of dimensions. Let Q, R be two n-dimensional boxes of volumes q and r, respectively. Divide each of them into 2^n sub-boxes, by dividing each of their n edges into two. In each of Q and R, number the sub-boxes by the binary vectors in $\{0,1\}^n$ in the natural way. For each $0 \leqslant k \leqslant n$, the kth level in each of Q and R is the set of sub-boxes with k 1's in their index (for example, the 1st level for $n = 2$ consists of the rectangles numbered by 01 and 10). Suppose that, for each $0 \leqslant k \leqslant n$, the kth level in R contains a sub-box whose volume is larger than or equal to the volume of every sub-box in the kth level of Q. Our inequality then says that $q \leqslant r$. Theorem 2.1 states this formally. Write

$$A_k = \{\phi \in \{0,1\}^n \colon \phi \text{ contains } k \text{ 1's}\}.$$

Let $\boldsymbol{\alpha} = (\alpha^1, \alpha^2, \ldots, \alpha^n)$ be a vector of functions α^i from the set $\{0, 1\}$ into the real numbers, and let $\boldsymbol{\phi} = (\phi_1, \phi_2, \ldots, \phi_n)$ be a vector in $\{0,1\}^n$. We then write $\alpha_\phi = \prod_{1 \leqslant i \leqslant n} \alpha^i(\phi_i)$.

Theorem 2.1. *Let $\boldsymbol{\alpha} = (\alpha^1, \alpha^2, \ldots, \alpha^n)$ and $\boldsymbol{\beta} = (\beta^1, \beta^2, \ldots, \beta^n)$ be two vectors of functions from $\{0,1\}$ into the nonnegative reals. Suppose that for each $0 \leqslant k \leqslant n$ there exists $\phi = \phi_k \in A_k$* such that

$$\alpha_\theta \leqslant \beta_\phi \quad \text{for every } \theta \in A_k. \tag{2.3}$$

Then $\prod_{1 \leqslant i \leqslant n}(\alpha^i(0)+\alpha^i(1)) \leqslant \prod_{1 \leqslant i \leqslant n}(\beta^i(0)+\beta^i(1))$.
Moreover, for all $0 \leqslant k \leqslant n$,

$$\sum\{\alpha_\phi\colon \phi \in \Lambda_k\} \leqslant \sum\{\beta_\phi\colon \phi \in \Lambda_k\}.$$

The proof uses the notion of *majorization*. For a vector $\boldsymbol{x} \in \mathbb{R}^n$ write $\boldsymbol{x}_{[j]}$ for the coordinate of $\boldsymbol{x}$ which is jth in size. That is, if $\boldsymbol{x} = (x_1, x_2, \dots, x_n)$ and the order on the coordinates is $x_{i_1} \geqslant x_{i_2} \cdots \geqslant x_{i_n}$, then $x_{[j]} = x_{i_j}$. We say that a vector $\boldsymbol{x}$ *majorizes* the vector $\boldsymbol{y}$ (written $\boldsymbol{y} \prec \boldsymbol{x}$) if

$$\sum_{j=1}^{k} x_{[j]} \geqslant \sum_{j=1}^{k} y_{[j]} \quad \text{for all } 1 \leqslant k \leqslant n$$

and

$$\sum_{i=1}^{n} x_{[i]} = \sum_{i=1}^{n} y_{[i]}.$$

A basic fact about majorization is the inequality quoted in [3] as the Karamata inequality (see also [8, Ch. 3, Proposition C1]).

Theorem 2.2. *If* $\boldsymbol{y} \prec \boldsymbol{x}$ *and* g *is a convex function, then*

$$\sum_{i=1}^{n} g(x_i) \geqslant \sum_{i=1}^{n} g(y_i).$$

For a vector $\boldsymbol{x} = (x_1, \dots, x_n)$ and $1 \leqslant k \leqslant n$ write $\boldsymbol{x}^{(k)}$ for the $\binom{n}{k}$-dimensional vector whose entries are $x_S = \sum\{x_i\colon i \in S\}$ for every $S \in \binom{[n]}{k}$. We shall need the following lemma, which is undoubtedly well known.

Lemma 2.3. *If* $\boldsymbol{y} \prec \boldsymbol{x}$ *then* $\boldsymbol{y}^{(k)} \prec \boldsymbol{x}^{(k)}$.

Proof. We may assume that $\boldsymbol{x}$ and $\boldsymbol{y}$ are arranged so that $x_i = x_{[i]}$ and $y_i = y_{[i]}$ $(1 \leqslant i \leqslant n)$. Arrange the sets in $\binom{[n]}{k}$ as $S_1, S_2, \dots, S_{\binom{n}{k}}$ so that $y_{S_1} \geqslant y_{S_2} \cdots \geqslant y_{S_{\binom{n}{k}}}$. It clearly suffices to show that, for each $1 \leqslant t \leqslant \binom{n}{k}$,

$$\sum_{1 \leqslant i \leqslant t} x_{S_i} \geqslant \sum_{1 \leqslant i \leqslant t} y_{S_i}. \tag{2.4}$$

Now, the family $\mathscr{F} = \{S_i\colon 1 \leqslant i \leqslant t\}$ is shifted, that is: if $S \in \mathscr{F}$, $1 \leqslant u < v \leqslant n$ and $v \in S$ and $u \notin S$ then $S - v + u \in \mathscr{F}$. Hence, the sequence $d_{\mathscr{F}}(v) = |\{S \in \mathscr{F}\colon v \in S\}|$ is descending. This, in turn, implies that there exist $m_1, m_2, \dots, m_r$ such that

$$\sum_{1 \leqslant i \leqslant t} y_{S_i} = \sum_{k=1}^{r}\left(\sum_{j=1}^{m_k} y_j\right) \quad \text{and} \quad \sum_{1 \leqslant i \leqslant t} x_{S_i} = \sum_{k=1}^{r}\left(\sum_{j=1}^{m_k} x_j\right).$$

Since for each k we have $\sum_{j=1}^{m_k} x_j \geqslant \sum_{j=1}^{m_k} y_j$, (2.4) follows. $\square$

Proof of Theorem 2.1. By continuity arguments we may assume that β^i and α^i are strictly positive. For each $1 \leqslant i \leqslant n$, let $a_i = \alpha^i(1)/\alpha^i(0)$ and $b_i = \beta^i(1)/\beta^i(0)$. Also write

$$p = \prod_{i=1}^{n} \alpha^i(0), \qquad q = \prod_{i=1}^{n} \beta^i(0).$$

Without loss of generality, assume that $b_1 \geqslant b_2 \cdots \geqslant b_n$. Write χ_k (where $0 \leqslant k \leqslant n$) for the vector $(1, 1, \dots, 1, 0, \dots, 0)$ whose first k entries are 1 and the rest are 0. Then, for every $0 \leqslant k \leqslant n$ and every $\psi \in \Lambda_k$ we have

$$\beta_\psi \leqslant q \prod_{j=1}^{k} b_j = \beta_{\chi_k}.$$

We may thus assume that ϕ_k in the assumption of the theorem is, in fact, equal to χ_k $(0 \leqslant k \leqslant n)$. This implies that the only inequality in (2.3), which involves $\beta^1(0)$ is the inequality for the 0th level, i.e., $p \leqslant q$. Hence, we may reduce $\beta^1(0)$ (thereby increasing b_1) until $p = q$, while all other inequalities in (2.3) remain valid. Similarly, the only inequality involving $\beta^n(1)$ is the last one, and hence we may reduce $\beta^n(1)$ until $\alpha_{\chi_n} = \beta_{\chi_n}$, still keeping (2.3). For each $1 \leqslant k \leqslant n$ the maximum of α_ϕ over $\phi \in \Lambda_k$ equals $p\prod_{j=1}^{k} a_{[j]}$. Hence, (2.3) (together with the assumptions that $p = q$ and $\alpha_{\chi_n} = \beta_{\chi_n}$) implies that $\prod_{j=1}^{k} b_j \geqslant \prod_{j=1}^{k} a_{[j]}$ for every $1 \leqslant k \leqslant n$, and $\prod_{j=1}^{n} b_j = \prod_{j=1}^{n} a_j$. Writing $x_i = \log_2 b_i$ and $y_i = \log_2 a_i$ we see that the vector $\boldsymbol{x} = (x_1, \dots, x_n)$ majorizes the vector $\boldsymbol{y} = (y_1, \dots, y_n)$. Applying Theorem 2.2 to the vectors $\boldsymbol{x}^{(k)}$ and $\boldsymbol{y}^{(k)}$ $(1 \leqslant k \leqslant n)$ with the function $g(x) = 2^x$, we obtain

$$B_k := \sum_{S \in \binom{[n]}{k}} \prod_{i \in S} b_i \geqslant \sum_{S \in \binom{[n]}{k}} \prod_{i \in S} a_i =: A_k.$$

The theorem now follows from the fact that

$$\sum\{\alpha_\phi\colon \phi \in \Lambda_k\} = pA_k, \qquad \sum\{\beta_\phi\colon \phi \in \Lambda_k\} = pB_k,$$

$$\prod_{i=1}^{n}(\alpha^i(0) + \alpha^i(1)) = \sum_{k \leqslant n} \sum_{\phi \in \Lambda_k} \alpha_\phi$$

and

$$\prod_{i=1}^{n}(\beta^i(0) + \beta^i(1)) = \sum_{k \leqslant n} \sum_{\phi \in \Lambda_k} \beta_\phi.$$

3. A proof of the n functions theorem (Theorem 1.1)

It should be noted first that since every finite distributive lattice can be embedded in some set lattice, it suffices to prove the theorem for $\mathscr{L}$ the set lattice of $\mathscr{P}([m])$.

Another observation is that by replacing each function α^k by the function $\underline{\alpha}^k$ defined by

$$\underline{\alpha}^k(A) = \begin{cases} \alpha^k(A), & A \in \mathscr{A}_k, \\ 0, & A \notin \mathscr{A}_k, \end{cases}$$

and similarly for β^k, we may assume that all $\mathscr{A}_k$ and $\mathscr{B}_k$ are equal to the entire power set $\mathscr{P}([m])$.

The proof is by induction on m.

Consider first the case $m = 0$. Then $\mathscr{P}([m]) = \{\emptyset\}$, and (1.3) says that

$$\prod_{1 \leqslant i \leqslant n} \alpha^i(\emptyset) \leqslant \prod_{1 \leqslant i \leqslant n} \beta^i(\emptyset).$$

In (1.4), if one of the $\mathscr{A}_i$'s is $\emptyset$, then both sides of the inequality are 0. If all $\mathscr{A}_i$'s are $\{\emptyset\}$, then (1.3) is just (1.4).

The case $m = 1$ will serve us later in the proof, so we prove it separately.

For $m = 1$, $\mathscr{P}([1]) = \{\phi, [1]\}$. The elements of the lattice can be represented by $0\,(= \phi)$ and $1\,(= [1])$. In this notation it is evident that given a binary vector (of lattice elements) $\boldsymbol{A} = (A_1, \ldots, A_n)$,

$$f_k(\boldsymbol{A}) = \begin{cases} 1, & |\{i\colon 1 = A_i\}| \geqslant k, \\ 0, & \text{otherwise}, \end{cases}$$

and so condition (1.3) can be expressed as

$$\alpha_\theta \leqslant \beta_{\chi_k} \quad \text{for every } \theta \in A_k, \quad 0 \leqslant k \leqslant n.$$

Theorem 2.1 then yields

$$\prod_{1 \leqslant i \leqslant n} (\alpha^i(0) + \alpha^i(1)) \leqslant \prod_{1 \leqslant i \leqslant n} (\beta^i(0) + \beta^i(1))$$

which can be written as

$$\prod_{1 \leqslant i \leqslant n} \alpha^i(\mathscr{L}) \leqslant \prod_{1 \leqslant i \leqslant n} \beta^i(\mathscr{L}),$$

which is the desired result. Assume now that the theorem holds for $m - 1$. Define functions

$$\begin{aligned} \hat{\alpha}^i(A) &= \alpha^i(A) + \alpha^i(A + m) \\ \hat{\beta}^i(A) &= \beta^i(A) + \beta^i(A + m) \end{aligned} \quad (i = 1, \ldots, n). \tag{3.1}$$

(Here $A \subseteq [m - 1]$ and $A + m$ denotes $A \cup \{m\}$.)

Lemma 3.1. *$\hat{\alpha}^i$, $\hat{\beta}^i$ satisfy condition* (1.3) *on* $\mathscr{P}([m - 1])$.

Proof. Let $\boldsymbol{A} = (A_1, \ldots, A_n)$, where $A_i \in \mathscr{P}([m - 1])$. Let $\mathscr{Q} = \mathscr{P}([1])$ and define $\tilde{\alpha}^i$, $\tilde{\beta}^i$ on $\mathscr{Q}$ as follows:

$$\begin{aligned} \tilde{\alpha}^i(\phi) &= \alpha^i(A_i), \\ \tilde{\alpha}^i([1]) &= \alpha^i(A_i + m) \\ \tilde{\beta}^i(\phi) &= \beta^i(f_i(\boldsymbol{A})) \\ \tilde{\beta}^i([1]) &= \beta^i(f_i(\boldsymbol{A}) + m) \end{aligned} \qquad (1 \leqslant i \leqslant n).$$

We claim that $\tilde{\alpha}^i$, $\tilde{\beta}^i$ satisfy (1.3) on $\mathscr{Q}$. As mentioned earlier, we have to show that

$$\tilde{\alpha}_\theta \leqslant \tilde{\beta}_{\chi_k} \quad (\theta \in A_k,\ 0 \leqslant k \leqslant n).$$

Indeed, fix k and let $\theta \in A_k$, define

$$B_i = \begin{cases} A_i & \theta(i) = 0, \\ A_i + m, & \theta(i) = 1. \end{cases}$$

Then

$$f_i(\boldsymbol{B}) = f_i(\boldsymbol{A}) + m, \quad 1 \leqslant i \leqslant k,$$

$$f_i(\boldsymbol{B}) = f_i(\boldsymbol{A}), \quad k + 1 \leqslant i \leqslant n.$$

Hence,

$$\begin{aligned} \tilde{\alpha}_\theta &= \prod_{\theta(i)=0} \alpha^i(A_i) \prod_{\theta(i)=1} \alpha^i(A_i + m) = \prod_{1 \leqslant i \leqslant n} \alpha^i(B_i) \\ &\leqslant \prod_{1 \leqslant i \leqslant n} \beta^i(f_i(\boldsymbol{B})) = \prod_{1 \leqslant i \leqslant k} \beta^i(f_i(\boldsymbol{A}) + m)) \prod_{k+1 \leqslant i \leqslant n} \beta^i(f_i(\boldsymbol{A})) = \tilde{\beta}_{\chi_k} \end{aligned}$$

(by (1.3)).

By the basis of the induction we have

$$\prod_{i=1}^n \tilde{\alpha}^i(Q) \leqslant \prod_{i=1}^n \tilde{\beta}^i(Q)$$

or

$$\prod_{i=1}^n \hat{\alpha}^i(A_i) \leqslant \prod_{i=1}^n \hat{\beta}^i(f_i(\boldsymbol{A})),$$

which proves the lemma. $\square$

Using the induction argument once more, this time for $\hat{\alpha}^i$, $\hat{\beta}^i$ on $\mathscr{P}([m - 1])$, we get

$$\prod_{i=1}^n \hat{\alpha}^i(\mathscr{P}([m - 1])) \leqslant \prod_{i=1}^n \hat{\beta}^i(\mathscr{P}([m - 1]))$$

but since $\hat{\alpha}^i(\mathscr{P}([m - 1])) = \alpha^i(\mathscr{P}([m]))$ and $\hat{\beta}^i(\mathscr{P}([m - 1])) = \beta^i(\mathscr{P}([m]))$ we get

$$\prod_{i=1}^n \alpha^i(\mathscr{P}([m])) \leqslant \prod_{i=1}^n \beta^i(\mathscr{P}([m]))$$

which proves the theorem.

Remark 1. The inductive step follows the original proof of Ahlswede and Daykin [1]. One can derive Theorem 1.1 from Theorem 2.1 also via Daykin's product theorem [5].

Remark 2. An important case, as in the Ahlswede–Daykin theorem, is that in which $\alpha^i \equiv 1 \equiv \beta^i$, which yields

$$\prod_{i=1}^{n} |\mathscr{A}_i| \leqslant \prod_{i=1}^{n} |f_i(\mathscr{A})|, \tag{3.2}$$

where $\mathscr{A} = (\mathscr{A}_1, \ldots, \mathscr{A}_n)$ is a vector of families of sets. A simple case of equality in (3.2) is where

$$\mathscr{A}_i = \{A\colon [i] \subset A \subset [m]\}, \quad 1 \leqslant i \leqslant n \ (n \leqslant m).$$

Here we have $f_k(\mathscr{A}) = \mathscr{A}_{n+1-k}$.

4. Extensions and possible extensions

Let us return for a moment to the geometrical interpretation of Theorem 2.1. Is it possible to generalize it to the case where each edge is divided into more than two parts? We conjecture that the answer is positive, but can prove only part of it: we can prove it only under the additional assumption that each ϕ_k in the hypothesis is nonincreasing and then only the inequality between the total volumes, and not separately for the 'levels', as in Theorem 2.1. But first we should define our terms.

As usual, $[m]^n$ is the set of vectors of length n whose terms are taken from $[m]$. Two vectors $\phi, \psi \in [m]^n$ are in the same level if they are permutations of one another (in other words, a level is an orbit of $[m]^n$ under the action of S_n). The level of ϕ is denoted by $\Lambda(\phi)$. For $\phi \in [m]^n$ let $\vec{\phi}$ be its nonincreasing rearrangement. The aforementioned geometrical result, stated formally, is:

Theorem 4.1. *Let* $\boldsymbol{\alpha} = (\alpha^1, \ldots, \alpha^n)$ *and* $\boldsymbol{\beta} = (\beta^1, \ldots, \beta^n)$ *be two vectors of functions from* $[m]$ *into the nonnegative reals. If* $\alpha_\phi \leqslant \beta_{\vec{\phi}}$ *for every* $\phi \in [m]^n$, *then* $\prod_{i=1}^n \sum_{k=1}^m \alpha^i(k) \leqslant \prod_{i=1}^n \sum_{k=1}^m \beta^i(k)$.

Proof. Apply Theorem 1.1 to the linear lattice $[m]$, in which $1 < 2 \cdots < m$. Note that, when considering $\phi \in [m]^n$ as a vector of elements from this lattice, $f_i(\phi) = \phi_{[i]}$ $(1 \leqslant i \leqslant n)$, i.e. $(f_i(\phi)) = \vec{\phi}$. Thus, the assumption of the theorem yields condition (1.3), while its conclusion is (1.4). □

Conjecture 4.1(a). In Theorem 4.1 the conclusion holds also under the weaker assumption that each level contains a vector ϕ such that $\alpha_\theta \leqslant \beta_\phi$ for each θ in this level.

Conjecture 4.1(b). Under the hypothesis of the theorem (or, indeed, even under the weaker assumption of (a) above), for every level Λ:

$$\sum \{\alpha_\phi\colon \phi \in \Lambda\} \leqslant \sum \{\beta_\phi\colon \phi \in \Lambda\}.$$

Write $\Delta = (m, m-1, \ldots, 1)$. The special case $m = n$ and Λ the 'diagonal' level, i.e. the level of Δ, was conjectured also by Rinott and Saks [9, Conjecture 1.1] (although in quite a different setting, and some translation work is necessary). In fact, the two conjectures are equivalent.

Proposition 4.2. *If Conjecture* 4.1(b) *is true for* $m = n$ *and* $\Lambda = \Lambda(\Delta)$, *then it is true in general.*

Proof. Let us first prove the case $m = n$ and Λ being a general level. Let ϕ be the nonincreasing representative of Λ. Define $\hat{\alpha}^i, \hat{\beta}^i\colon [n] \to \mathbb{R}^+$ as follows:

$$\hat{\alpha}^i = \alpha^i \circ \phi \circ \Delta, \qquad \hat{\beta}^i = \beta^i \circ \phi \circ \Delta.$$

(As usual, $\circ$ stands for composition of functions.)

Let $\theta \in [n]^n$ and define $\tilde{\theta} = \phi \circ \Delta \circ \theta$. With this notation we have $\hat{\alpha}_\theta = \alpha_{\tilde{\theta}}$. By the hypothesis of Theorem 4.1, $\alpha_{\tilde{\theta}} \leqslant \beta_{\vec{\tilde{\theta}}}$, but since ϕ and Δ are nonincreasing, $\vec{\theta} = \tilde{\vec{\theta}}$. Combining all this with the definition of the functions $\hat{\beta}^i$, we get $\hat{\alpha}_\theta = \alpha_{\tilde{\theta}} \leqslant \beta_{\vec{\tilde{\theta}}} = \beta_{\tilde{\vec{\theta}}} = \hat{\beta}_{\vec{\theta}}$.

Thus, the functions $\hat{\alpha}^i$ and $\hat{\beta}^i$ satisfy the condition of Conjecture 4.1(b). Assuming that the conjecture is true for the diagonal level, we get that

$$\sum_{\pi \in S_n} \hat{\alpha}_\pi \leqslant \sum_{\pi \in S_n} \hat{\beta}_\pi. \tag{4.1}$$

But

$$\sum_{\pi \in S_n} \hat{\alpha}_\pi = \sum_{\pi \in S_n} \alpha_{\tilde{\pi}} = \sum_{\pi \in S_n} \alpha_{\phi \circ \Delta \circ \pi} = \sum_{\sigma \in S_n} \alpha_{\phi \circ \sigma}$$

$$= \sum_{\psi \in \Lambda(\phi)} \sum_{\phi \circ \sigma = \psi} \alpha_\psi = |\mathrm{Stab}(\phi)| \sum_{\psi \in \Lambda(\phi)} \alpha_\psi.$$

Similarly, $\sum_{\pi \in S_n} \hat{\beta}_\pi = |\mathrm{Stab}(\phi)| = \sum_{\psi \in \Lambda(\phi)} \beta_\psi$, hence, by (4.1),

$$\sum_{\psi \in \Lambda(\phi)} \alpha_\psi \leqslant \sum_{\psi \in \Lambda(\phi)} \beta_\psi.$$

The case $m > n$ follows easily from the case $m = n$, since a vector in $[m]^n$ contains at most n different elements, and thus we may restrict the problem to a subset of $[m]$ of size n. The case $m < n$ is reduced to the case $m = n$ by adding coordinates on which α^i and β^i are zero.

The proof of Theorem 4.1 indicates that a natural approach to Conjecture 4.1(b) is via Theorem 1.1. Indeed, Conjecture 4.1(b) is a special case of a possible strengthening of Theorem 1.1. To state it, we need the notion of 'levels' for n-tuples in lattices. Let $X = (X_1, \dots, X_n)$ be a vector of elements of a distributive lattice $\mathscr{L}$, where $X_1 \geqslant X_2 \cdots \geqslant X_n$. The *level* of X, denoted by $\Lambda(X)$, is the set of vectors $A = (A_1, \dots, A_n)$ of elements of $\mathscr{L}$ such that $f_i(A) = X_i$ $(1 \leqslant i \leqslant n)$. Notice that for the linear lattice $[m]$ this definition coincides with the one given above.

Conjecture 4.3. If α^i, β^i $(1 \leqslant i \leqslant n)$ are as in Theorem 1.1 and they satisfy condition (1.3), then for every level Λ:

$$\sum_{\bar{A} \in \Lambda} \prod_{i=1}^{n} \alpha^i(A_i) \leqslant \sum_{\bar{A} \in \Lambda} \prod_{i=1}^{n} \beta^i(A_i).$$

This is stronger than Theorem 1.1, since summing up for all levels yields (1.4) for $\mathscr{A}_1 = \mathscr{A}_2 = \cdots = \mathscr{A}_n = \mathscr{L}$, which, as noted in the proof of Theorem 1.1, suffices for the general proof of (1.4).

The original proof of Daykin's inequality [4] ($n = 2$, $\alpha^i \equiv \beta^i \equiv 1$) yielded, in fact, Conjecture 4.3 for this case. In fact, it is not hard to prove Conjecture 4.3 for $n = 2$ and general α^i, β^i. This follows from general results of Ahlswede and Daykin [2]. For completeness we give here a short proof. Let us remark that both this and the proof of [2] use Theorem 1.1, while Daykin's proof [4] is direct.

Proposition 4.4. *Conjecture* 4.3 *is true for* $n = 2$.

Proof. As usual, it suffices to consider the lattice $\mathscr{L}$ of subsets of a given set, say $[m]$. Let Λ be a level defined by $(A, B) \in \Lambda$ if $A \cup B = X$, $A \cap B = Y$. For $Y \subseteq A \subseteq X$ write $\tilde{A} = Y \cup (X \setminus A)$. Define functions $\tilde{\alpha}^i$, $\tilde{\beta}^i$ $(i = 1, 2)$ by

$$\tilde{\alpha}^i(A) = \begin{cases} \alpha^i(A)\alpha^{3-i}(\tilde{A}), & Y \subseteq A \subseteq X, \\ 0, & \text{otherwise,} \end{cases}$$

$$\tilde{\beta}^i(A) = \begin{cases} \beta^i(A)\beta^{3-i}(\tilde{A}), & Y \subseteq A \subseteq X, \\ 0, & \text{otherwise.} \end{cases}$$

We shall show that $\tilde{\alpha}^i$, $\tilde{\beta}^i$ satisfy (1.3). Let (A, B) be a pair of sets. We may assume that $Y \subseteq A$, $B \subseteq X$, or else the left-hand side of (1.3) is zero. We then have

$$\begin{aligned} \tilde{\alpha}^1(A)\tilde{\alpha}^2(B) &= \alpha^1(A)\alpha^2(B)\alpha^2(\tilde{A})\alpha^1(\tilde{B}) \\ &\leqslant \beta^1(A \cup B)\beta^2(A \cap B)\beta^1(\tilde{A} \cup \tilde{B})\beta^2(\tilde{A} \cap \tilde{B}) = \tilde{\beta}^1(A \cup B)\tilde{\beta}^2(A \cap B) \end{aligned}$$

(the inequality follows from (1.3) on α^i, β^i; the last equality from De Morgan laws). By the AD inequality we have

$$\tilde{\alpha}^1(\mathscr{L})\tilde{\alpha}^2(\mathscr{L}) \leqslant \tilde{\beta}^1(\mathscr{L})\tilde{\beta}^2(\mathscr{L}). \tag{4.2}$$

But

$$\tilde{\alpha}^i(\mathscr{L}) = \sum_{(A,B) \in \Lambda} \alpha^i(A)\alpha^{3-i}(B) \quad \text{and} \quad \tilde{\beta}^i(\mathscr{L}) = \sum_{(A,B) \in \Lambda} \beta^i(A)\beta^{3-i}(B), \ (i = 1, 2).$$

Thus, (4.2) yields

$$\left(\sum_{(A,B) \in \Lambda} \alpha^1(A)\alpha^2(B)\right)^2 \leqslant \left(\sum_{(A,B) \in \Lambda} \beta^1(A)\beta^2(B)\right)^2,$$

which proves the proposition. □

It is possible to formulate Conjecture 4.3 in terms of polynomials. Let α^i, β^i be as above, and define polynomials

$$a^i(x) = \sum_{S \subseteq [m]} \alpha^i(S) x_S$$

and

$$b^i(x) = \sum_{S \subseteq [m]} \beta^i(S) x_S \quad (i \leqslant n),$$

where x is a vector of indeterminates $(x_1, \dots, x_n)$, and $x_S = \prod_{j \in S} x_j$ for each $S \subseteq [m]$. Also let $A(x) = \prod_{i \leqslant n} a^i(x)$, $B(x) = \prod_{i \leqslant n} b^i(x)$. Then Theorem 2.1 states that if α^i, β^i satisfy (2.3), then $A(1, \dots, 1) \leqslant B(1, \dots, 1)$. In fact, by replacing α^i by functions $\tilde{\alpha}^i$ defined by $\tilde{\alpha}^i(S) = \alpha^i(S)x_S$ and similarly for β^i, it follows by Theorem 2.1 that

$$A(x) \leqslant B(x) \tag{4.3}$$

for all nonnegative vectors x. Conjecture 4.3 strengthens this by hypothesizing that (4.3) holds coefficientwise, i.e. that each coefficient of A is not larger than the corresponding coefficient in B.

Another approach to Conjecture 4.1(b) is via majorization. This, like some of the previous conjectures, was also conjectured by Rinott and Saks [10]:

Conjecture 4.1(c). Under the same conditions as in Conjecture 4.1(b)

$$(\log \alpha_\pi)_{\pi \in S_n} \prec_w (\log \beta_\pi)_{\pi \in S_n}.$$

($x \prec_w y$ denotes the fact that y weakly majorizes x, i.e., the condition $\sum x_i = \sum y_i$ is dropped from the majorization conditions.)

By more ad hoc methods we have proved the corresponding log-majorization result which yields the case $m = n = 3$ of Conjecture 4.1(a).

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