Krein's Strings, the Symmetric Moment Problem, and Extending a Real Positive Definite Function

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Abstract

The symmetric moment problem is to find a possibly unique, positive symmetric measure that will produce a given sequence of moments $\{M_n\}$. Let us assume that the (Hankel) condition for existence of a solution is satisfied, and let σ_n be the unique measure, supported on n points, whose first $2n$ moments agree with $M_0, \dots, M_{2n-1}$. It is known that $\sigma_{2n} \Longrightarrow \sigma_0$ (weak convergence) and $\sigma_{2n+1} \Longrightarrow \sigma_\infty$, where σ_0 and σ_∞ are solutions to the full moment problem. Moreover, $\sigma_0 = \sigma_\infty$ if and only if the problem has a unique solution. In this paper we present an analogue of this theorem for Krein's problem of extending to $\mathbb{R}$ a real, even positive definite function originally defined on $[-T, T]$ where $T < \infty$. Our proof relies on the machinery of Krein's strings. As we show, these strings help explain the connection between the moment and the extension problems.

1 Introduction

In the (Hamburger) classical moment problem, one is given a sequence of real numbers $\{M_k\}$ and asked to verify the existence and uniqueness of σ, a positive measure on $\mathbb{R}$, such that for all $k \geq 0$, $M_k = \int \gamma^k \, d\sigma(\gamma)$. It is known that a necessary and sufficient condition for the existence of σ is that for any n the Hankel matrix $H_n = (h_{ij} \stackrel{d}{=} M_{i+j} : 0 \leq i, j \leq n)$ is positive definite (see, e.g., [2]). The uniqueness is, generally, a subtler question. However, for the symmetric moment problem (where $M_{2n+1} = 0$ for all n) there is a "hands on" way to settle the uniqueness question while approximating at least one solution. This method is based on the fact that for a given sequence of $2n$ moments $\{M_k : k = 0, \dots, 2n-1\}$ there exists a unique positive measure σ_n, supported on n points, such that $\int \gamma^k \, d\sigma = M_k$ for $k = 0, \dots, 2n-1$ [2, p. 31]. One can then prove the following known result:

THEOREM 1.1 (Even-Odd Theorem) *$\sigma_{2n} \Longrightarrow \sigma_0$ and $\sigma_{2n+1} \Longrightarrow \sigma_\infty$ (weak convergence of measures), where the symmetric positive measures σ_0 and σ_∞ are solutions to the full symmetric moment problem. Moreover, σ_0 and σ_∞ are the only symmetric solutions σ with dense polynomials in $L^2(\sigma)$, and $\sigma_0 = \sigma_\infty$ if and only if this moment problem has an overall unique solution.*

This theorem can be proved by reducing the symmetric moment problem to a moment problem on the half-line, where one can use Stieltjes results. Alternatively, in [7, section 5.5] we provide an outline of the proof following Akheizer's

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reasoning in [2]. In [11] Simon provides a somewhat different proof, while an additional approach is outlined in Section 4. We show that an analogue of this even-odd theorem applies to the problem of extending a real, even positive definite function.

A complex-valued function f defined on an interval $[-T,T]$ is a positive definite function if $\sum_{i,j=1}^{n} f(t_i - t_j)\xi_i\xi_j^* \geq 0$ for all n, $0 \leq t_i \leq T$, and $\xi_i \in \mathbb{C}$. In [8] Krein studies the problem of extending a Hermitian positive definite function from its initial domain $[-T,T]$ to a Hermitian positive definite function on $\mathbb{R}$. Approaching this question in an analogous way to the classical moment problem, Kreĭn affirms the existence of such an extension and provides a certain necessary and sufficient condition for its uniqueness. Note that according to Bochner's theorem we can identify any extension F of f as the Fourier transform of a positive measure σ (denoted as $F = \hat{\sigma}$ or $\sigma = \check{F}$).

Suppose now that R is a *real*, even positive definite function defined on $[-T,T]$. With $\delta_n \stackrel{d}{=} T/(n-1)$, the sequence $\{c_k \stackrel{d}{=} R(k\delta_n) : k = -n+1,\dots,n-1\}$ is a positive definite sequence, i.e., $\sum_{i,j=0}^{n-1} c_{i-j}\xi_i\xi_j^* \geq 0$ for all $\xi_i \in \mathbb{C}$. Therefore, there exists a unique *symmetric* measure μ_n that is supported on at most n points in $[-\pi,\pi]$ so that $c_k = \int e^{i\omega k}\, d\mu_n(\omega)$ (see, e.g., [1]). Let $\sigma_n(\omega) \stackrel{d}{=} \mu_n(\omega/\delta_n)$, and let the *dimension* of an even positive definite function be the cardinality of the support of the corresponding measure. Note that $R_n(t) \stackrel{d}{=} \int e^{it\omega}\, d\sigma_n(\omega) = \hat{\sigma}_n(t)$ is the unique positive definite function of dimension $n, \tilde{R}$, for which $\tilde{R}(k\delta_n) = R(k\delta_n)$ for $k = 0,1,\dots,n-1$.

Our main result is the following:

THEOREM 1.2 *Suppose R is a real, even positive definite function on $[-T,T]$. Then $R_{2n} \longrightarrow R_0$ and $R_{2n+1} \longrightarrow R_\infty$ uniformly on compact subsets, where R_0 and R_∞ are positive definite extensions of R. Moreover, R has a unique extension overall if and only if $R_0 = R_\infty$. Finally, $\sigma_0 \stackrel{d}{=} \check{R}_0$ and $\sigma_\infty \stackrel{d}{=} \check{R}_\infty$ are the only symmetric positive measures σ for which $\{e^{it\omega} : |t| \leq T/2\}$ span $L^2(\sigma)$.*

We prove this theorem in Section 5 using Krein's theory of vibrations of generalized strings. In Section 2 you will find a brief introduction to this theory that is essentially adopted from Dym and McKean's monograph on the subject [5]. In Section 3 we give further details of a few results that are essential to proving this theorem. We prove those statements for which either the proof or the explicit statement could not be found in the literature.

The fundamental property of these strings is that they are in perfect equivalence with real positive definite functions on $\mathbb{R}$. It will be shown that all the strings that correspond to positive definite extensions of a real positive definite function R defined on $[-T,T]$ share a nontrivial common string (Claim 3.3). Moreover, this shared string, "tied" with $k = 0$ and $k = \infty$, corresponds to R_0 and R_∞, respectively, from Theorem 1.2.

Krein's strings can also explain the apparent analogy between the even-odd theorem (Theorem 1.1) and Theorem 1.2, as both are essentially questions of extending the aforementioned shared string. This analogy will become clearer in Section 4, where the connection between the moment problem and strings is examined. For now, we note the following:

- The classical moment problem can be regarded as an infinitesimal version of the extension problem. Namely, if σ is a solution to the moment problem, then $R \stackrel{d}{=} \hat{\sigma}$ is a positive definite function and $M_k = i^k R^{(k)}(0)$. Thus, the classical moment problem is equivalent to the problem of finding a positive definite function R with given derivatives at 0.
- Conversely, the extension problem can be regarded as a question of finding a positive definite function with given finite differences. Indeed, recall that in solving the extension problem, as described in Theorem 1.2, we look for an n-dimensional positive definite function R_n such that $R_n(k\delta) = R(k\delta)$ for $k = 0, \dots, n-1$. For a given δ, let Δ be the finite difference operator defined by

$$(\Delta f)(x) \stackrel{d}{=} \frac{f(x+\frac{\delta}{2}) - f(x-\frac{\delta}{2})}{\delta}.$$

  Then, for any symmetric real function f, $\Delta^{2k+1} f(0) = 0$ and

$$\Delta^{2k} f(0) = \frac{1}{\delta^{2k}} \left[\binom{2k}{k} f(0) + 2 \sum_{j=0}^{k-1} \binom{2k}{j} (-1)^j f((k-j)\delta) \right],$$

  so $\{\Delta^{2k} f(0) : k \le n-1\}$ is in perfect equivalence with $\{f(k\delta) : 0 \le k \le n-1\}$. In particular, computing R_n amounts to finding a positive definite function with given finite differences.
- Furthermore, finding R_n in the extension problem is equivalent to solving a (finite) symmetric moment problem: Let σ_n be, as before, $\check{R}_n$. Then with $\delta = \delta_n$,

$$\begin{aligned} \Delta^{2k} R(0) &= \frac{1}{\delta^{2k}} \sum_{j=0}^{2k} \binom{2k}{j} (-1)^j R((k-j)\delta) \\ &= \frac{1}{\delta^{2k}} \int \sum \left[\binom{2k}{j} (-1)^j e^{-ij\delta\theta} \right] e^{ik\delta\theta} \, d\sigma_n(\theta) \\ &= \int \left(\frac{e^{i\delta\theta/2} - e^{-i\delta\theta/2}}{\delta} \right)^{2k} d\sigma_n(\theta) \\ &= (-1)^k \int_{-\pi/\delta}^{\pi/\delta} \left(\frac{\sin(\theta\delta/2)}{\delta/2} \right)^{2k} d\sigma_n(\theta). \end{aligned} \tag{1.1}$$

Define $\bar{\sigma}_n$ as the measure induced by σ_n on $[-2/\delta, 2/\delta]$ through the change of variable $2/\delta \sin(\theta\delta/2) \mapsto \gamma$; then

$$(-1)^k \Delta^{2k} R(0) = \int \gamma^{2k}\, d\bar{\sigma}_n .$$

Thus, finding R_n is essentially equivalent to finding the measure $\bar{\sigma}_n$ with the corresponding moments. In particular, $\bar{\sigma}_n$ is the unique symmetric measure supported on at most n points in $[-2/\delta, 2/\delta]$ with the even moments $(-1)^k \Delta^{2k} R(0)$ for $k = 0, \ldots, n-1$.

This paper ends with Section 6, where we comment on the actual computations involved in finding $\bar{\sigma}_n$ and where a numerical example of Theorem 1.2 is worked out. This example also underscores the interaction between the moment and the extension problem.

2 A Review of Krein's Strings

Let Δ be a positive symmetric measure on $\mathbb{R}$ with $\int \frac{d\Delta(\omega)}{1+\omega^2} < \infty$. In a series of papers (1952–1954, an account of which is given in [5]),[1] Krein showed that there is an interesting connection between vibrations of a generalized string and certain subspaces of $\boldsymbol{Z}(\Delta) \stackrel{d}{=} L^2(d\Delta)$. A string is characterized by

- $m(x)$, its positive, nondecreasing, right-continuous (accumulated) mass function, defined for $x \geq 0$,
- its length $l \stackrel{d}{=} \inf\{x \geq 0 : m(x) = m(\infty)\}$, and
- a tying constant, $0 \leq k \leq \infty$, is specified if $l + m(l) < \infty$ ("short string"). If $k < \infty$, then the string is tied at $x = l + k$, while if $k = \infty$, it is loose at $x = l$. The other end of the string, $x = 0$, is always free.

Krein showed that there is a one-to-one correspondence between positive measures Δ with $\int \frac{d\Delta(\omega)}{1+\omega^2} < \infty$ and such strings. The basic idea is that on the appropriate domain in $L^2(dm)$, the second-order differential operator $G = d^2/dm\,dx$ is a nonpositive self-adjoint operator and therefore has a spectral representation.

To be more precise, let $A_\gamma(x) = A(x,\gamma)$ be the eigenfunctions of G (or if you will, the eigenvibrations of the string): $GA_\gamma(x) = -\gamma^2 A_\gamma(x)$ subject to the boundary conditions $A_\gamma(0) = 1$ and $A_\gamma^-(0) = 0$ ($A_\gamma^-(x)$ is the derivative from the left with respect to x). For $dm(x) = dx$, i.e., the "ordinary string," we have $A_\gamma(x) = \cos(\gamma x)$.

The even transform for $f \in L^2(dm)$ is defined by

$$\tilde{f}(\gamma) = \int_{0^-}^{l} f(x) A(x,\gamma)\, dm(x) .$$

The *principal spectral function* of the string is the unique symmetric and positive measure Δ for which

[1] The notations and interpretations here are borrowed from [5].

- the even transform is an isometry *onto* the subspace of even functions in $\boldsymbol{Z}(\Delta)$

$$\|f\|_m^2 = \int_{0^-}^{l} |f|^2\, dm(x) = \frac{1}{\pi}\int |\tilde{f}|^2\, d\Delta = \frac{1}{\pi}\|\tilde{f}\|_\Delta^2,$$

- there is an inverse transform

$$f(x) = \frac{1}{\pi}\int A(x,\gamma)\,\tilde{f}(\gamma)d\Delta(\gamma),$$

- Δ is a spectral measure: The domain of G is the class of all $f \in L^2(dm)$ such that $\int \gamma^4 \tilde{f}(\gamma)^2\, d\Delta(\gamma) < \infty$ and $\widetilde{Gf} = -\gamma^2 \tilde{f}(\gamma)$.

For $m(x) = x$, we have $d\Delta(\gamma) = d\gamma$ and $\tilde{f}$ is the cosine transform.

Similarly, one defines an odd transform (analogous to the sine transform) based on the function $B(x,\gamma) = -A^+(x,\gamma)/\gamma$ (with $A^+(x,\gamma)$ being the derivative from the right with respect to x): Let $\boldsymbol{X}$ be the subspace of $L^2([0,l+k),dx)$ comprised of functions that are constant on intervals where the string has no mass. The odd transform is defined for $f \in \boldsymbol{X}$ as

$$\check{f}(\gamma) = \int_0^{l+k} f(x)B(x,\gamma)dx.$$

The odd transform is an invertible isometry onto the subspace of odd functions in $\boldsymbol{Z}(\Delta)$; more precisely,

- for $f \in \boldsymbol{X}$, $\check{f}$ is an odd function in $\boldsymbol{Z}(\Delta)$ and

$$\|f\|_{\boldsymbol{X}}^2 = \int_0^{l+k} |f|^2\, dx = \frac{1}{\pi}\int |\check{f}|^2\, d\Delta = \frac{1}{\pi}\|\check{f}\|_\Delta^2,$$

- for any odd $g \in \boldsymbol{Z}(\Delta)$ there exists a unique $f \in \boldsymbol{X}$ such that $g = \check{f}$ and

$$f(x) = \frac{1}{\pi}\int g(\gamma)B(x,\gamma)\, d\Delta(\gamma).$$

Remarks.

- By their construction, for any $\gamma \in \mathbb{C}$, $A(x_0,\gamma)$ and $B(x_0,\gamma)$ depend only on $m(x)$ for $x \le x_0$. Furthermore, they are holomorphic functions in γ of exponential type $T = \int_0^{x_0} \sqrt{m'(s)}\, ds$ [5, chapter 6].
- Since we have a positive definite function in mind, we require $\int d\Delta < \infty$. This simplifies matters; in particular, it means that all our strings start with a discrete mass at 0 ($m[0] > 0$).

A symmetric positive measure Δ is *a* spectral function of the string (m,l) if the even transform, $f \mapsto \tilde{f}$, defines an isometry of $L^2(dm)$ into $\boldsymbol{Z}(\Delta)$. In particular, the principal spectral function is a spectral function. Krein proves that any such Δ is a principal spectral function of a longer string $S_\Delta = (m_\Delta, l_\Delta, k_\Delta)$ that contains the original one; i.e., $m(x) = m_\Delta(x)$ for $x \in [0,l]$ [5, section 5.10]. In particular, a long string has a unique spectral function: its principal one.

3 Basic Results and Consequences

The first result we need does not involve strings. Let $\boldsymbol{Z}^T(\Delta) \stackrel{d}{=} \mathrm{Span}\{e^{it\bullet} : t \leq T\} \subset \boldsymbol{Z}(\Delta)$.

CLAIM 3.1 Let R be a positive definite function defined on $[-2T, 2T]$, and Ω the set of all positive measures that extend R. Then either

- there is more than one extension, and there exists $\Delta \in \Omega$ such that $\boldsymbol{Z}^T(\Delta) \subsetneq \boldsymbol{Z}(\Delta)$ or
- there is only one extension, and $\boldsymbol{Z}^T(\Delta) = \boldsymbol{Z}(\Delta)$.

Remark. An analogous statement holds for the moment problem, with $\boldsymbol{Z}^T$ replaced by $\boldsymbol{Z}_\infty$ (the span of the polynomials in $\boldsymbol{Z}(\Delta)$).

PROOF: The first alternative follows immediately from Krein's work [8]; the proof goes along the lines of the corresponding proof for the moment problem. Alternatively, it can easily be derived directly: Suppose $\Delta_1 \neq \Delta_2$ are in Ω. If $\boldsymbol{Z}^T(\Delta_i) \neq \boldsymbol{Z}(\Delta_i)$ for either $i = 1, 2$, then we are done. Otherwise, with $\Delta \stackrel{d}{=} \frac{1}{2}\Delta_1 + \frac{1}{2}\Delta_2 \in \Omega$, one can verify that $\boldsymbol{Z}^T(\Delta) \subsetneq \boldsymbol{Z}(\Delta)$. Furthermore, if R is real on $[-T, T]$, then by considering $(d\Delta(\gamma) + d\Delta(-\gamma))/2$, we can assume Δ is symmetric. Note that Ω, being convex, either is a singleton or is infinite.

On the other hand, if the extension is unique, it follows from Krein's analysis[2] that for all $z \in \mathbb{C} \setminus \mathbb{R}$, $(u-z)^{-1} \in \boldsymbol{Z}^T(\Delta)$. We continue following Akheizer's proof of a similar statement in the moment problem [2, section 2.3.2]: If $\boldsymbol{Z}^T(\Delta) \neq \boldsymbol{Z}(\Delta)$, then there exists $g \in \boldsymbol{Z}(\Delta)$ such that the functions

$$\phi(z) = \int \frac{g(u)}{u-z}\, d\Delta(u)\,, \qquad \psi(z) = \int \frac{g(u)^*}{u-z}\, d\Delta(u)\,,$$

are both 0 on $\mathbb{C}^+$. It follows that $g \equiv 0$. $\square$

By next restricting our attention to the real case (symmetric weight), we can use Krein strings to say more about this extension problem.

Consider the string $S = (m, l, k)$. For a fixed γ, $A(x, \gamma)$ is a continuous function of x, and for a fixed $x < l$ it is a holomorphic function of γ. Similarly, $B(x, \gamma)$ is constant in x on mass-free intervals of the string and is holomorphic in γ. It follows that for a fixed $a < l$ and any $\varphi \in L^2(dm)$ and $\psi \in \boldsymbol{X}$,

$$F(\gamma) = \int_{0-}^{a-} \varphi(x) A(x, \gamma)\, dm(x) + \int_0^a \psi(x) B(x, \gamma)\, dx \tag{3.1}$$

is a holomorphic function. Let K^{a-} be the space of holomorphic functions F that can be expressed as above, i.e., $F = \tilde{\varphi} + \breve{\psi}$. Note that the definition of K^{a-} depends

[2] In most of these papers, Krein does not provide proofs; at most he gives a hint as to how the proof goes.

only on the short string $S_a \overset{d}{=} (l=a, m(x)1_{x<a})$. Let Γ be the set of spectral measures of S_a. Then by the isometric properties of the odd and even transforms, for any $\Delta \in \Gamma$,

$$\frac{1}{\pi}\int |F(\gamma)|^2\, d\Delta(\gamma) = \int_{0-}^{a-} |\varphi(x)|^2\, dm(x) + \int_0^a |\psi(x)|^2\, dx < \infty\,.$$

Thus, the restriction of any $F \in K^{a-}$ to $\mathbb{R}$ yields a unique function in $\boldsymbol{Z}(\Delta)$, or $K^{a-} \subset \boldsymbol{Z}(\Delta)$. Conversely, we have the following:

CLAIM 3.2 There are exactly two measures in Γ such that $K^{a-} \supset \boldsymbol{Z}(\Delta)$, meaning any $f \in \boldsymbol{Z}(\Delta)$ is a restriction of $F \in K^{a-}$ to Δ. These measures Δ_0 and Δ_∞ are the principal spectral functions of the strings obtained by tying S_a (without adding any mass) with $k=0$ and $k=\infty$, respectively.

PROOF: Let $\Delta \in \Gamma$ have a string $(m_\Delta, l_\Delta, k_\Delta)$ that has an extra piece of mass so that $m_\Delta(l_\Delta) > m_\Delta(a-)$. Then there exists a nontrivial $\varphi \in L^2(dm_\Delta)$ that is supported on $[a, l_\Delta]$. Of course, φ is orthogonal in $L^2(dm_\Delta)$ to any $\tilde{\varphi}$ that is supported on $[0,a)$; thus, with $f|_\Delta$ denoting f restricted to the support of Δ,

$$F(\gamma) \overset{d}{=} \int_{0-}^{l_\Delta} \varphi(x) A_\Delta(x,\gamma)\, dm_\Delta(x)\bigg|_\Delta \neq 0\,,$$

is orthogonal to K^{a-} in $\boldsymbol{Z}(\Delta)$, and $K^{a-} \not\supset \boldsymbol{Z}(\Delta)$.

Next, suppose $\Delta \in \Gamma$ has a string $(m_\Delta, l_\Delta, k_\Delta)$ with $l_\Delta = a$ and $m_\Delta(l_\Delta) = m(a-)$, but $0 < k_\Delta < \infty$. Let $\varphi \overset{d}{=} 1_{(a, a+k_\Delta)}$. Then $\varphi \in \boldsymbol{X}_\Delta$, and

$$F(\gamma) \overset{d}{=} \int_0^{l_\Delta} \varphi(x) B_\Delta(x,\gamma)\, dx\bigg|_\Delta \neq 0\,,$$

and, as before, F is orthogonal to K^{a-} in $\boldsymbol{Z}(\Delta)$. Indeed, in this case the codimension of K^{a-} in $\boldsymbol{Z}(\Delta)$ is 1.

Finally, any $f \in \boldsymbol{Z}(\Delta_{0/\infty})$ has its even inverse transform supported on $[0,a)$, since this is the support of their corresponding strings. Similarly, the odd transform has to be supported on $[0,a)$ since it is constant on $[a, a+k]$. Thus $K^{a-} \supset \boldsymbol{Z}(\Delta_{0/\infty})$. $\square$

Let $x_0(T) \le x_1(T)$ be the minimal and maximal solutions, respectively, of

$$T = \int_0^x \sqrt{m'(s)}\, ds\,. \tag{3.2}$$

By using strings, Krein claims that if $\boldsymbol{Z}^T(\Delta) \neq \boldsymbol{Z}(\Delta)$, then $x_0 < \infty$ and $\boldsymbol{Z}^T(\Delta) = K^{x_0-}$, meaning that there is an equivalence between holomorphic functions in K^{x_0-} and their restrictions to the support of Δ, which, as the claim goes, lie in

$\boldsymbol{Z}^T(\Delta)$ [9]. A proof of this identification, which will be used extensively in Claim 3.3 below, can be found in [5, section 6.4].

Let us go back to the case of a real positive definite function R that is given on $[-2T, 2T]$. As before, let Ω be the nonempty set of its extensions (real or Hermitian) to the whole line. If $|\Omega| = 1$, then there is a unique string (short or long) associated with the (symmetric) positive measure Δ, and $\boldsymbol{Z}^T(\Delta) = \boldsymbol{Z}(\Delta)$. Conversely, we have the following:

CLAIM 3.3 If the extension is not unique, then there exists a short string $S_c = (m_c, l_c)$ such that a symmetric σ is in Ω if and only if σ is the principal spectral function of a longer string $(m_\sigma, l_\sigma, k_\sigma)$ with $m_\sigma(x) = m_c(x)$ for all $x < l_c$.

PROOF: Claim 3.1 guarantees the existence of a symmetric $\Delta \in \Omega$ for $\boldsymbol{Z}^T(\Delta) \neq \boldsymbol{Z}(\Delta)$. Let x_0 be the minimal solution of (3.2) and note that $x_0 < \infty$ since $\boldsymbol{Z}^T(\Delta) \neq \boldsymbol{Z}(\Delta)$. The string S_c we look for is defined by the mass function

$$m_c(x) \stackrel{d}{=} \begin{cases} m_\Delta(x)\,, & x < x_0\,, \\ m_\Delta(x_0-)\,, & x \geq x_0\,, \end{cases}$$

and, clearly, $l_c = x_0$. Let σ be a symmetric measure in Ω. We need to show that σ is a spectral function of the string S_c. This will be established if we can prove that for any $\varphi \in L^2(dm_c)$, $F(\gamma) \stackrel{d}{=} \int_{0-}^{x_0-} \varphi(x) A_c(x,\gamma) dm_c(x)$ satisfies $\|F\|_{\boldsymbol{Z}(\sigma)} = \|F\|_{\boldsymbol{Z}(\Delta)}$. Since $K^{x_0}|_\Delta = \boldsymbol{Z}^T(\Delta)$, $F|_\Delta \in \boldsymbol{Z}^T(\Delta)$, and therefore there exist trigonometric polynomials $p(\gamma) = \sum \alpha_k e^{it_k\gamma}$ with $|t_k| \leq T$ such that $p_n \to F$ in $\boldsymbol{Z}(\Delta)$. Since both σ and Δ are in Ω, $\|p_n\|_{\boldsymbol{Z}(\sigma)} = \|p_n\|_{\boldsymbol{Z}(\Delta)}$, so proving that $p_n \to F$ in $\boldsymbol{Z}(\sigma)$ will complete the proof. But $\{p_n\}$ is a Cauchy sequence in $\boldsymbol{Z}(\Delta)$ and thus also in $\boldsymbol{Z}(\sigma)$, and it remains to show only that the limit is F.

Since $p_n \in \boldsymbol{Z}^T(\Delta) = K^{x_0}$, there exist $\varphi_n \in L^2(dm_c)$ and $\psi_n \in \boldsymbol{X}_c$ such that

$$p_n(\gamma) = \int_{0-}^{x_0-} \varphi_n(x) A_c(x,\gamma) dm_c(x) + \int_0^{x_0} \psi_n(x) B_c(x,\gamma) dx\,.$$

Hence,

$$\begin{aligned} |p_n(\gamma) - F(\gamma)| &= \left| \int_{0-}^{x_0-} \big(\varphi_n(x) - \varphi(x)\big) A_c(x,\gamma) dm_c(x) \right. \\ &\qquad \left. + \int_0^{x_0} \psi_n(x) B_c(x,\gamma)\, dx \right| \\ &\leq \|\varphi_n - \varphi\|_{L^2(dm_c)} \left[\int_{0-}^{x_0-} |A_c(x,\gamma)|^2\, dm_c(x) \right]^{1/2} \\ &\qquad + \|\psi_n - 0\|_{L^2([0,x_0],dx)} \left[\int_0^{x_0} |B_c(x,\gamma)|^2\, dx \right]^{1/2}\,. \end{aligned}$$

Thus, with

$$C_A(\gamma) \overset{d}{=} \left[\int_{0-}^{x_0-} |A_c(x,\gamma)|^2\, dm_c(x)\right]^{1/2} < \infty$$

and

$$C_B(\gamma) \overset{d}{=} \left[\int_{0}^{x_0} |B_c(x,\gamma)|^2\, dx\right]^{1/2} < \infty\,,$$

we have

$$|p_n(\gamma) - F(\gamma)| \le \frac{\sqrt{2}}{\sqrt{\pi}} \max(C_A, C_B) \|p_n - F\|_{\boldsymbol{Z}(\Delta)}\,.$$

Hence $p_n(\gamma) \longrightarrow F(\gamma)$ for every $\gamma \in \mathbb{C}$, and it follows that p_n converges to F in $\boldsymbol{Z}(\sigma)$.

As for the second half of the claim, suppose that the symmetric measure σ is a principal spectral function of a string (m,l,k) that agrees with S_c for $x < l_c$. Since for a fixed $|t| \le T$, $\cos(t\gamma) \in \boldsymbol{Z}^T(\Delta)$, there exists $\varphi_t \in L^2(dm_c)$ such that

$$\cos(t\gamma) = \int_{0-}^{l_c-} \varphi_t(x) A_c(x,\gamma) dm_c(x)\,.$$

For $x < l_c$, $m(x) = m_c(x)$, thus, $A_m(x,\gamma) = A_c(x,\gamma)$ for any $x < l_c$ and $\gamma \in \mathbb{C}$ (similarly, $B_m(x,\gamma) = B_c(x,\gamma)$). Hence the even transform of $\varphi_t(x)$, from $L^2(dm)$ into $L^2(\sigma)$, is $\cos(t\gamma)$ as well, or

$$\varphi_t(x) = \int \cos(t\gamma) A_m(x,\gamma)\, d\sigma(\gamma)\,.$$

It follows that

$$\begin{aligned}\int \cos(t\gamma)\cos(s\gamma) d\sigma(\gamma) &= \int_{0-}^{l_c-} \varphi_t(x)\varphi_s(x) dm_c(x)\\ &= \int \cos(t\gamma)\cos(s\gamma) d\Delta(\gamma)\,.\end{aligned}$$

An analogous treatment for $\sin(t\gamma)$ using the odd transforms completes the proof that $\sigma \in \Omega$. ☐

Remark 3.4. So far we learned that either the extension problem is determined, in which case there exists a unique extension prescribed by the symmetric measure Δ and $\boldsymbol{Z}^T(\Delta) = \boldsymbol{Z}(\Delta)$, or it is not determined, in which case there are exactly two extensions, Δ_0 and Δ_∞, for which $\boldsymbol{Z}^T(\Delta) = \boldsymbol{Z}(\Delta)$ (cf. Claim 3.2). It can be shown that in the second case, these two measures are supported on the zeros of $A(l,\cdot)$ and $B(l,\cdot)$, respectively. The latter are holomorphic functions of type $\le T$ with interlacing zeros [5, section 6.3]. In particular, Δ_0 and Δ_∞ are discrete and, as in the moment case, have disjoint support (with $\Delta_0[0] = 0 < \Delta_\infty[0]$).

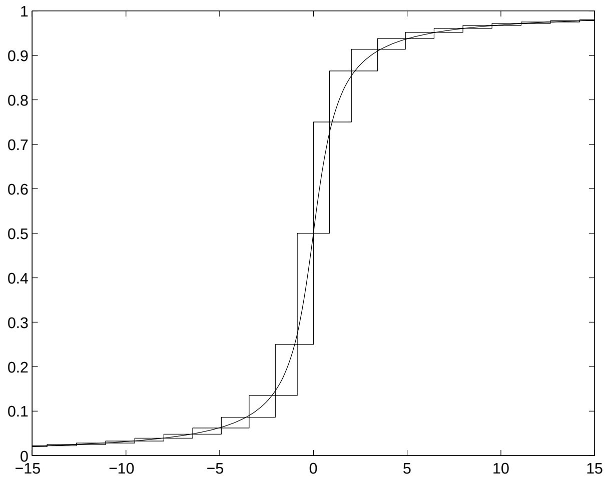


FIGURE 3.1. Example 3.5: Δ_0, Δ_∞, and Δ. The three distributions whose graphs are plotted here are spectral measures of the short string S_c that corresponds to $R(r) = e^{-|r|}$ on $[-2, 2]$. The continuous line is of $\Delta(\gamma) = \frac{1}{\pi}\frac{1}{1+\gamma^2}\,d\gamma$, while Δ_0 and Δ_∞ are the principal spectral functions of S_c tied with $k = 0$ and $k = \infty$, respectively. Figure 3.2 depicts $R_0 = \hat{\Delta}_0$ and $R_\infty = \hat{\Delta}_\infty$.

As Krein points out in [10], one can gain some insight into the relation between short strings and "short" positive definite functions as follows: Suppose that a force impulse (a delta function) is applied to the left end of the string ($x = 0$) at time $t = 0$. The motion of that end of the string as a function of time is given by $\psi(t) = \int \sin(\omega t)/\omega\, d\Delta(\omega)$. Note that $\psi'(t) = \int \cos(\omega t)\, d\Delta(\omega) = \hat{\Delta}(t)$. Let $T(x) \stackrel{d}{=} \int_0^x \sqrt{m'(y)}dy$. Then T is the travel time of the wave from 0 to x. Since the time it takes the wave to reach the point x and return to 0 is $2T(x)$, it is intuitively clear that for $t < 2T(x)$, $\psi(t)$ depends only on $m|_{[0,x)}$. In other words, any string that agrees with $m|_{[0,x)}$ will reproduce the same motion, $\psi(t)$, for $t \leq 2T(x)$, and therefore $R(t) = \hat{\Delta}(t)$ will be identical for all such strings.

EXAMPLE 3.5 Consider the positive definite function $R(r) = e^{-|r|}$ for $r \in [-2, 2]$. R extends naturally to $\mathbb{R}$ as $e^{-|r|}$, which corresponds to the measure $d\Delta(\gamma) = \frac{1}{\pi}(1+\gamma^2)^{-1}d\gamma$. Since $\int \log \Delta'(\gamma)(1+\gamma^2)^{-1}d\gamma > -\infty$, it follows by Szegö's alternative [5, section 4.2] that this extension is not unique.

Applying [5, rule 1, p. 265] to [5, example 6.10.1] yields the common string S_c mentioned in Claim 3.3:

$$m_c(x) = \pi + \pi^2 x, \qquad l_c = \frac{1}{\pi}.$$

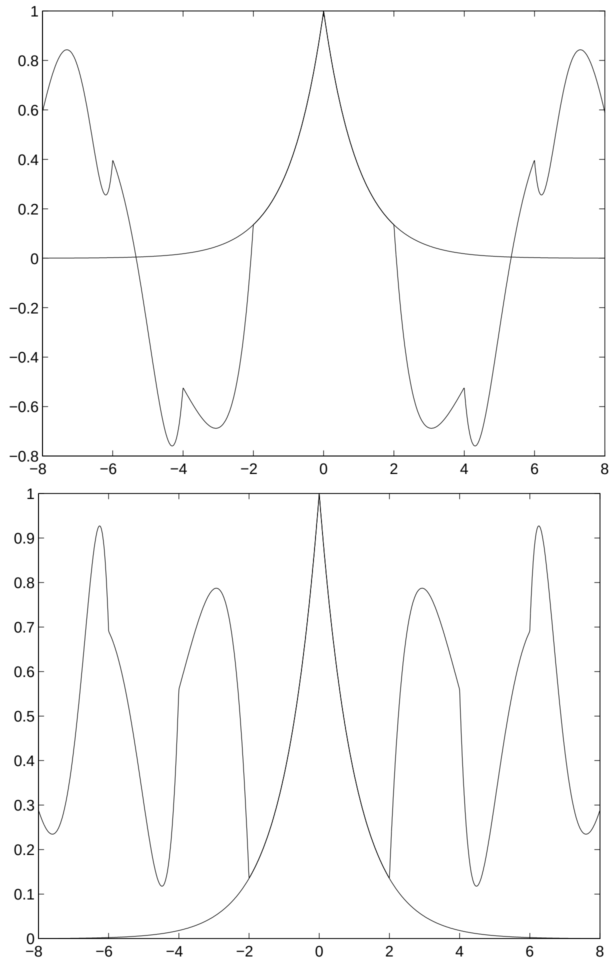


FIGURE 3.2. Example 3.5: Extensions of $e^{-|r|}$ from $[-2,2]$. The upper picture is of $R_0 = \hat{\Delta}_0$ and $R(r) = e^{-|r|}$, which is plotted for reference. The two positive definite functions are the same on $[-2,2]$. The lower picture contrasts $R_\infty = \hat{\Delta}_\infty$ and R.

Also provided this way are

$$A(x,\gamma) = \cos(\pi\gamma x) - \gamma \sin(\pi\gamma x)\,, \qquad B(x,\gamma) = -\pi \sin(\pi\gamma x) - \pi\gamma \cos(\pi\gamma x)\,.$$

The supports of Δ_0 and Δ_∞, the principal spectral measures of S_c tied with $k = 0$ and $k = \infty$, respectively, can be found by solving the equations $A(l_c,\gamma) = 0$ and $B(l_c,\gamma) = 0$, respectively. The weights at $\pm\gamma_n$ can then be found by computing $\frac{1}{2}\pi\|A(\bullet,\gamma_n)\|_m^{-2}$ [5, section 5.5]. Figure 3.1 depicts Δ_0, Δ_∞, and Δ, while Figure 3.2 describes the corresponding positive definite functions R_0, R_∞, and R, respectively.

The following remarks about strings and their spectral functions will be required later on:

Remark 3.6. Since the inverse even transform is an isometry, by applying it to the indicator function $1_{\{0\}}(\gamma) \in \mathbb{Z}(\Delta)$, we learn that if $\Delta[0] > 0$, then $m_\Delta(l_\Delta) = \pi/\Delta[0]$ (cf. [5, exercise 5.8.4]).

Remark 3.7. Suppose that the Δ_n satisfy $\Delta_n[0] > \varepsilon > 0$, $\Delta_n(\mathbb{R}) = 1$, and $\Delta_n \Longrightarrow \Delta$. Then, by the previous remark, the corresponding mass functions m_n are uniformly bounded. Thus, there exists a subsequence $\{n_k\}$ and a nondecreasing right-continuous function $m(x)$ such that $m_{n_k}(x) \to m(x)$ for any x such that $m[x] = 0$. Since the Δ_n are finite measures, $m[0] > 0$, and one can then follow the rest of Dym and McKean's proof of an analogous case [5, p. 203] to prove that m has to be the mass function of the string of Δ. Therefore, $m_n(x) \to m_\Delta(x)$ for all x such that $m_\Delta[x] = 0$.

Remark 3.8. By applying the odd transform to the function $\gamma \mapsto (\gamma + ib)^{-1}$ and then letting $b \to 0$, we see that [5, p. 192, (4)]

$$\frac{1}{\pi}\int \gamma^{-2}\, d\Delta(\gamma) = \begin{cases} l + k & \text{if } l + m(l) < \infty, \\ l & \text{if } l + m(l) = \infty. \end{cases}$$

4 Strings and the Moment Problem

Strings can be applied just as effectively to the study of the symmetric moment problem. Assume Δ is a symmetric measure with moments of all orders: $M_n = \int \gamma^n\, d\Delta(\gamma) < \infty$. Let $d \le \infty$ be the number of growth points of Δ, or the dimension of R. From [5, sections 5.8–5.9], we learn the following:

- If $d = 2n + 2 < \infty$, then the mass function of the string consists of masses m_i located at $0 = x_0 < x_1 < \cdots < x_n = l$. The string is tied at $x_{n+1} > l$, i.e., $k = x_{n+1} - l > 0$.
- If $d = 2n + 1$, then m and l are as above, but the string is tied at $l = x_n$ with $k = \infty$.
- If $d = \infty$, then $m(x)$ begins with an infinite number of isolated jumps: m_i located at $0 = x_0 < x_1 < x_2 \cdots$. Let $L \stackrel{d}{=} \sup_k x_k$; then $m(L) + L = \infty$ if and only if the moment problem is determinate. If it is not determinate, then any spectral function of the short string (m, L) is a solution of this symmetric moment problem, and these are the only symmetric solutions. In this case, tying the string with $k = 0$ and $k = \infty$ will produce the only two symmetric solutions, Δ_0 and Δ_∞, respectively, with dense polynomials. This can be deduced from Claim 3.2 after we (readily) identify $K^{L-} = \boldsymbol{Z}_\infty$.

It can also be shown that any spectral function of a string of the type described above has moments of all orders.

The precise recipe for recovering the string from the moments is taken from [5, section 5.8]: Orthogonalize the powers $\{\gamma^n : 0 \leq n < d\}$ forming a sequence of alternatively even and odd polynomials $\{P_n(\gamma)\}_{n<d}$, normalized so that

$$P_{2n}(0) = 1\,, \quad 2n < d\,,$$
$$\frac{1}{\pi}\int \frac{P_{2n+1}(\gamma)}{\gamma}\, d\Delta(\gamma) = 1\,, \quad 2n+1 < d\,.$$

The masses m_n and spacings $x_{n+1} - x_n$ are specified by

$$\begin{aligned} m_n &= \frac{\pi}{\|P_{2n}\|_\Delta^2}\,, \qquad 2n < d\,, \\ x_{n+1} - x_n &= \frac{\pi}{\|P_{2n+1}\|_\Delta^2}\,, \quad 2n+1 < d\,. \end{aligned} \tag{4.1}$$

Remark 4.1. An immediate corollary of (4.1) is that x_k is expressed wholly in terms of $M_0, M_2, \ldots, M_{4k-2}$ and that m_k depends only on $M_0, M_2, \ldots, M_{4k}$.

Strings can be used to prove the even-odd theorem in a way that sheds some extra light. Let S_c be the string constructed from the given moments; i.e., m_c consists of a sequence of jumps m_i located at x_i as described above. Let σ_{2n} be the unique measure supported on $2n$ points such that

$$M_k = \int \gamma^k\, d\sigma_{2n}(\gamma)\,, \quad k = 0, 1, \ldots, 4n-1\,.$$

The string S_{2n} associated with σ_{2n} consists of masses $m_0^{2n}, m_1^{2n}, \ldots, m_{n-1}^{2n}$ located at $0 = x_0^{2n} < x_1^{2n} < \cdots < x_{n-1}^{2n}$ and is tied at x_n^{2n}. By Remark 4.1, $x_i^{2n} = x_i$ for $0 \leq i \leq n$, and similarly $m_i^{2n} = m_i$ for $0 \leq i \leq n-1$. Thus S_{2n} is a "substring" of S_c. It is now obvious that the strings S_{2n} converge pointwise to the string S_c. Furthermore, if S_c is short, then $k_{2n} = x_n - x_{n-1} \underset{n}{\longrightarrow} 0$ and *intuitively* it follows that $\sigma_{2n} \Longrightarrow \Delta_0$.

Similarly, the string S_{2n+1}, corresponding to σ_{2n+1}, satisfies $x_i^{2n+1} = x_i$ and $m_i^{2n+1} = m_i$ for $0 \leq i \leq n$ and is tied with $k_{2n+1} = \infty$. Therefore, S_{2n+1} converges to S_c, and obviously $k_{2n+1} \longrightarrow \infty$, so we should expect $\sigma_{2n+1} \Longrightarrow \Delta_\infty$. This intuition can be turned into a rigorous proof quite easily; however, since a similar task is ahead of us, we will skip it here.

5 Proof of the Main Theorem

Let $\mathfrak{M}$ be the set of finite symmetric positive measures on $\mathbb{R}$. Let $\delta = \delta_n = T/(n-1)$ and $\Delta f(x) = \big(f(x+\delta/2) - f(x-\delta/2)\big)/\delta$. Given R, a real positive definite function on $[-T, T]$, we define

$$\Sigma \stackrel{d}{=} \{\sigma \in \mathfrak{M} : \hat{\sigma}(t) = R(t) \text{ for all } t \in [-T, T]\}\,,$$

$$\Sigma_n \stackrel{d}{=} \{\sigma \in \mathfrak{M} : \hat{\sigma}(k\delta) = R(k\delta) \text{ for } k = 0, 1, \ldots, n-1\}\,,$$

$$\bar{\Sigma}_n \stackrel{d}{=} \left\{ \bar{\sigma} \in \mathfrak{M} : \operatorname{supp}(\bar{\sigma}) \subset [-2/\delta, 2/\delta] \text{ and for } k = 0, 1, \ldots, n-1, \right.$$
$$\left. \int \gamma^{2k} \, d\bar{\sigma} = (-1)^k \Delta^{2k} R(0) \right\}.$$

Let $\varphi_\delta(\theta) \stackrel{d}{=} 2/\delta \sin(\theta\delta/2)$, and for any $\sigma \in \Sigma_n$; define the measure $\bar{\sigma}_\delta \stackrel{d}{=} \sigma \circ \varphi_\delta^{-1}$; i.e., $\bar{\sigma}_\delta$ is the pullback of σ by φ_δ. Clearly, $\bar{\sigma}_\delta$ is supported on $[-2/\delta, 2/\delta]$, and since as in (1.1) for $k = 0, 1, \ldots, n-1$,

$$\int \left(\frac{\sin(\theta\delta/2)}{\delta/2} \right)^{2k} d\sigma(\theta) = (-1)^k \Delta^{2k} R(0),$$

it follows that $\bar{\sigma}_\delta \in \bar{\Sigma}_n$. In particular, for σ_n, the unique measure in Σ_n that is supported on at most n points (in $[-\pi/\delta, \pi/\delta]$), $\bar{\sigma}_n \stackrel{d}{=} \sigma \circ \varphi_\delta^{-1} \in \bar{\Sigma}_n$. Furthermore, since $\bar{\sigma}_n$ is the unique measure in $\bar{\Sigma}_n$ that is supported on at most n points, its string, $m_{\bar{\sigma}_n}$, is a substring of $m_{\bar{\sigma}}$ for any $\bar{\sigma} \in \bar{\Sigma}_n$, i.e., $m_{\bar{\sigma}_n}(x) = m_{\bar{\sigma}}(x)$ for all $x \le l_{\bar{\sigma}_n}$.

Remark 5.1. As you recall, we denoted by μ_n the unique symmetric positive measure supported on at most n points in $[-\pi, \pi]$ for which $R(k\delta) = \int e^{ik\omega} \, d\mu_n(\omega)$ for $k = 0, 1, \ldots, n-1$. It can be shown that if the sequence $\{R(k\delta) : k = 0, 1, \ldots, n-1\}$ is strictly positive definite, then μ_n is supported on exactly n points [1, chapter 1]. Since σ_n is determined from μ_n by the simple change of variable $\omega \mapsto \omega/\delta$, it follows that both σ_n and $\bar{\sigma}_n$ will be supported on exactly n points in this case. If R is not determined from $[-T, T]$, then the sequence $\{R(k\delta) : k = 0, 1, \ldots, n-1\}$ is strictly positive definite [8], and therefore $\bar{\sigma}_n$ is supported on exactly n points in the nondetermined case.

CLAIM 5.2 $\{\sigma_n\}$ is a tight set of measures, and any of its limit measures is in Σ.

PROOF: Let $\alpha = [\inf_{|t| \ge 1} (1 - \frac{\sin t}{t})]^{-1}$. Since for any $\rho > 0$

$$\sigma_n \left(\left[-\frac{1}{\rho}, \frac{1}{\rho} \right]^C \right) \le \frac{\alpha}{\rho} \int_0^\rho \big(1 - R_n(r)\big) \, dr$$

[3, proposition 8.29]; to prove tightness, it suffices to show that the R_n have a uniform modulus of continuity at $r = 0$.

Since σ_n is supported on $[-\pi/\delta_n, \pi/\delta_n]$, for $|r| \le \delta_n$,

$$\begin{aligned} |1 - R_n(r)| &= \int_{-\pi/\delta_n}^{\pi/\delta_n} (1 - \cos \omega r) \, d\sigma_n(\omega) = 2 \int_{-\pi/\delta_n}^{\pi/\delta_n} \sin^2 \frac{\omega r}{2} \, d\sigma_n(\omega) \\ &\le 2 \int_{-\pi/\delta_n}^{\pi/\delta_n} \sin^2 \frac{\omega \delta_n}{2} \, d\sigma_n(\omega) \qquad (5.1) \\ &= 1 - R_n(\delta_n) = 1 - R(\delta_n). \end{aligned}$$

For $\varepsilon > 0$, there exists a $\rho > 0$ such that

$$|r| < \rho \Longrightarrow |R(0) - R(r)| < \varepsilon. \qquad (5.2)$$

For any n big enough so that $\delta_n < \rho$ and for $|r| < \rho$, let $r_n = k\delta_n$ be the closest grid point to the left of r; i.e., $k \in \mathbb{N}$ and

$$0 \le r - r_n < \delta .$$

Note that $R_n(r_n) = R(r_n)$; thus with (5.2) and (5.1),

$$\begin{aligned}|1 - R_n(r)| &\le |1 - R_n(r_n)| + |R_n(r_n) - R_n(r)| \\ &\le |1 - R(r_n)| + \sqrt{2\big(1 - R_n(r - r_n)\big)} \le \varepsilon + \sqrt{2\varepsilon} .\end{aligned}$$

Finally, as

$$R(k\delta_n) = R_n(k\delta_n) , \quad |k| \le n - 1 ,$$

the limit has to be a real extension of R. $\square$

Remark 5.3. It is clear from the definitions that $\sigma_{n_k} \Longrightarrow \nu$ if and only if $\bar{\sigma}_{n_k} \Longrightarrow \nu$.

If R has a unique extension R_e, then the previous claim assures us that $R_n \to R_e$ uniformly on compact subsets, and by Claim 3.1, with $\sigma = \hat{R}_e$, $\mathbb{Z}^{T/2}(\sigma) = \mathbb{Z}(\sigma)$. This proves the theorem in the determined case.

Conversely, assume R is not determined from $[-T, T]$. Then all real extensions of R share a common short string, $S_c = (m_c, l_c)$ (Claim 3.3). Let σ_0 and σ_∞ be the principal spectral functions of this string tied with $k = 0$ and $k = \infty$, respectively. We first show that $\sigma_{2n+1} \Longrightarrow \sigma_\infty$. Let $\bar{\sigma}_\delta \overset{d}{=} \sigma_\infty \circ \varphi_\delta^{-1}$; then $\bar{\sigma}_\delta \in \bar{\Sigma}_n$ and $\bar{\sigma}_\delta[0] \ge \sigma_\infty[0] > 0$ (Remark 3.4); therefore $m_{\bar{\sigma}_\delta}(l_{\bar{\sigma}_\delta}) \le m_c(l_c)$ (Remark 3.6). As noted above, the string of $\bar{\sigma}_n$ is a substring of any other string whose principal spectral function is in $\bar{\Sigma}_n$. In particular, $m_{\bar{\sigma}_n}(l_{\bar{\sigma}_n}) \le m_{\bar{\sigma}_\delta}(l_{\bar{\sigma}_\delta})$, and since $\bar{\sigma}_{2n+1}[0] > 0$ (as $\bar{\sigma}_{2n+1}$ has exactly $2n+1$ points of increase), it follows that $\bar{\sigma}_{2n+1}[0] \ge \sigma_\infty[0]$. Clearly, $\bar{\sigma}_{2n+1}[0] = \sigma_{2n+1}[0]$; therefore, for any ν such that $\sigma_{2n_k+1} \Longrightarrow \nu$, $\nu[0] \ge \sigma_\infty[0]$, and it follows that $m_\nu(l_\nu) \le m_c(l_c)$. But $\nu \in \Sigma$, so we can conclude that $m_\nu \equiv m_c$ (and $l_\nu = l_c$). It remains to verify that $k_\nu = \infty$, which follows immediately from $\nu[0] > 0$ and Remark 3.8. Hence, $\nu = \sigma_\infty$ and Claim 5.2 completes the proof that $\sigma_{2n+1} \Longrightarrow \sigma_\infty$.

We conclude the proof of our theorem by showing next that $\bar{\sigma}_{2n} \Longrightarrow \sigma_0$ (cf. Remark 5.3). It suffices to show that $\underline{\lim}(l_{\bar{\sigma}_{2n}} + k_{\bar{\sigma}_{2n}}) \le l_c$. Indeed, suppose it so and assume that $\bar{\sigma}_{2n_k} \Longrightarrow \nu$. Then

$$\frac{1}{\pi}\int \gamma^{-2}\, d\nu(\gamma) \le \underline{\lim} \int \gamma^{-2}\, d\bar{\sigma}_{2n_k} = \underline{\lim}(l_{\bar{\sigma}_{2n}} + k_{\bar{\sigma}_{2n}}) \le l_c .$$

This implies that $l_\nu \le l_c$ (Remark 3.8), and since $\nu \in \Sigma$,

$$m_\nu(l_\nu) \le m_\nu(l_c) = m_c(l_c) < \infty .$$

We conclude that

$$l_\nu + k_\nu = \frac{1}{\pi}\int \gamma^{-2}\, d\nu(\gamma) \le l_c ,$$

and therefore $l_\nu = l_c$ and $k_\nu = 0$, or $\nu = \sigma_0$.

Let $S^{2T} = (m_{2T}, l_{2T}, k_{2T} = \infty)$ be an extension of the common short string $S_c = (m_c, l_c)$ such that $l_c < l_{2T} < \infty$ and $m'_{2T}(x)$ exists and is positive for any $x \in (l_c, l_{2T})$. Let σ^{2T} be the principal spectral function of S^{2T}. Let $R^{2T} = \hat{\sigma}^{2T}$, and consider the problem of extending R^{2T} from $[-2T, 2T]$. Our treatment thus far assures us that $\bar{\sigma}^{2T}_{2n-1} \Longrightarrow \sigma^{2T}$. As before, it is easy to see that $\bar{\sigma}^{2T}_{2n-1}[0] \geq \sigma^{2T}[0] > 0$ and therefore for any x with $m_{2T}[x] = 0$, $m_{\bar{\sigma}^{2T}_{2n-1}}(x) \to m_{2T}(x)$ (Remark 3.7). The simple identity $2T/[(4n-1)-1] = T/(2n-1)$ implies that $\sigma^{2T}_{4n-1} \in \Sigma_{2n}$ and therefore also that $\bar{\sigma}^{2T}_{4n-1} \in \bar{\Sigma}_{2n}$. Recall that $\bar{\sigma}_{2n} \in \bar{\Sigma}_{2n}$ is supported on exactly $2n$ points, so the string of any other measure $\bar{\sigma} \in \bar{\Sigma}_{2n}$ will satisfy $m_{\bar{\sigma}}(x) = m_{\bar{\sigma}_{2n}}(x)$ for any $x < l_{\bar{\sigma}_{2n}} + k_{\bar{\sigma}_{2n}}$ (cf. Section 4). It follows that for any x such that $m_{2T}[x] = 0$ and $x < \underline{\lim}(l_{\bar{\sigma}_{2n}} + k_{\bar{\sigma}_{2n}})$, $m_{\bar{\sigma}_{2n}}(x) \to m_{2T}(x)$. But $\bar{\sigma}_{2n}$ is independent of the particular extension S^{2T} and therefore $\underline{\lim}(l_{\bar{\sigma}_{2n}} + k_{\bar{\sigma}_{2n}}) \leq l_c$, which completes the proof.

6 A Word on the Computation

One way to compute σ_n is to find μ_n and then make the necessary change of variable (scaling). To compute μ_n, one can follow the prescription outlined in [1, chapter 1]. As to the complexity of this numerical procedure, it calls for the computation of three $n \times n$ determinants, the finding of a 1-dimensional null space of a certain $n \times n$ Toeplitz matrix, and the computation of the complex roots of a certain polynomial of degree n.

Alternatively, one can compute σ_n by solving the finite moment problem,

$$\int \gamma^n \, d\bar{\sigma}_n = (-1)^k \Delta^{2k} R(0) \, .$$

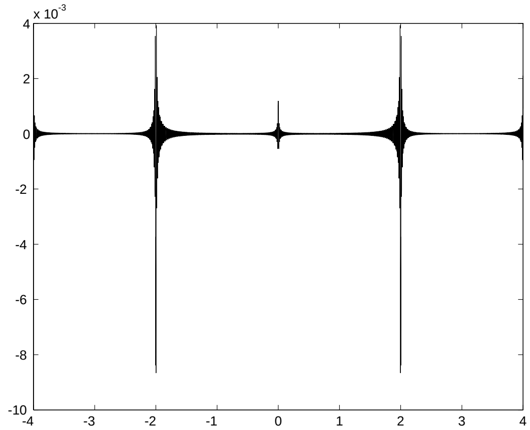
Ordinarily, this "moment approach" should not be the preferred solution since it is known that the map that takes you from the moments to the corresponding measure is numerically ill-conditioned [6]. Thus, in fact, the moment approach composes two numerically ill-conditioned maps, the first from R to the "moments" $\{(-1)^k \Delta^{2k} R(0) : k = 0, \ldots, n-1\}$, and the second from the moments to $\bar{\sigma}_n$. Nevertheless, one might gain something from the moment approach.

Typically, one would solve the moment problem by first computing the associated tridiagonal Jacobi matrix, or equivalently, find the coefficients in the three-term recursion formula of the orthogonal polynomials. Given the Jacobi matrix, the measure is easily recovered: It is supported on the eigenvalues of this tridiagonal matrix. As noted in [6], the ill-conditioned part is the computation of the Jacobi matrix. Thus, if we can somehow compute the $n \times n$ Jacobi matrix explicitly, then the moment approach will provide a faster and more accurate way to compute σ_n than by computing it through μ_n. We next demonstrate this by an example.

EXAMPLE 6.1 We continue with the study we began in Example 3.5 of extending $R(r) = e^{|-r|}$ from $[-2, 2]$. As noted there, the measure σ defined by $d\sigma(\omega) \stackrel{d}{=}$

TABLE 6.1. Example 6.1: Convergence of σ_{2n} vs. σ_{2n+1}. $R(r) = e^{-|r|}$ on $[-2,2]$. The table describes the numerical convergence of σ_{2n+1} and σ_{2n} to σ_∞ and σ_0, respectively. σ_n is supported on n points with a jump of $\rho^2_{n,i}$ at $\pm\omega_{n,i}$. The upper half of the table illustrates the convergence of σ_{2n}. The first two columns describe σ_0. You can check that $2\sum_{i=1}^{11}\rho_i^2 \sim$ 0.9807 (out of 1). Columns 3 through 5 specify the relative distance in the location of the jump between σ_0 and σ_n. The next three columns record the relative error in the size of the jump. The second half of the table does the same for σ_{2n+1}; here $\frac{1}{2} + 2\sum_{i=2}^{11}\rho_i^2 \sim 0.9798$. Note the approximate quadratic convergence.

σ_0		$\left\|\frac{\omega_i-\omega_{n,i}}{\omega_i}\right\|$ $(*10^{-3})$			$\left\|\frac{\rho_i^2-\rho_{n,i}^2}{\rho_i^2}\right\|$		
ω_i	ρ_i^2	$n=40$	$n=80$	$n=160$	$n=40$	$n=80$	$n=160$
0.8603	0.3649	0.1392	0.0339	0.0084	0.0003	0.0001	0.0000
3.4256	0.0728	0.2032	0.0495	0.0122	0.0029	0.0007	0.0002
6.4373	0.0230	0.2144	0.0522	0.0129	0.0095	0.0023	0.0006
9.5293	0.0108	0.2176	0.0529	0.0130	0.0206	0.0050	0.0012
12.6453	0.0062	0.2193	0.0532	0.0131	0.0362	0.0087	0.0021
15.7713	0.0040	0.2206	0.0533	0.0131	0.0568	0.0135	0.0033
18.9024	0.0028	0.2220	0.0535	0.0132	0.0825	0.0194	0.0048
22.0365	0.0021	0.2234	0.0536	0.0132	0.1140	0.0265	0.0065
25.1724	0.0016	0.2251	0.0537	0.0132	0.1516	0.0346	0.0084
28.3096	0.0012	0.2269	0.0538	0.0132	0.1962	0.0440	0.0107
31.4477	0.0010	0.2290	0.0539	0.0132	0.2485	0.0546	0.0132
σ_∞		$\left\|\frac{\omega_i-\omega_{n,i}}{\omega_i}\right\|$ $(*10^{-3})$			$\left\|\frac{\rho_i^2-\rho_{n,i}^2}{\rho_i^2}\right\|$		
ω_i	ρ_i^2	$n=41$	$n=81$	$n=161$	$n=41$	$n=81$	$n=161$
0	0.5000	0	0	0	0.0001	0.0000	0.0000
2.0288	0.3270	0.1743	0.0436	0.0109	0.0011	0.0003	0.0001
4.9132	0.0765	0.2005	0.0501	0.0125	0.0054	0.0014	0.0003
7.9787	0.0305	0.2057	0.0513	0.0128	0.0138	0.0034	0.0009
11.0855	0.0160	0.2077	0.0517	0.0129	0.0264	0.0065	0.0016
14.2074	0.0098	0.2090	0.0519	0.0130	0.0435	0.0107	0.0027
17.3364	0.0066	0.2102	0.0521	0.0130	0.0654	0.0159	0.0039
20.4692	0.0048	0.2115	0.0522	0.0130	0.0925	0.0222	0.0055
23.6043	0.0036	0.2129	0.0523	0.0130	0.1250	0.0296	0.0073
26.7409	0.0028	0.2145	0.0524	0.0130	0.1637	0.0382	0.0094
29.8786	0.0022	0.2162	0.0525	0.0130	0.2091	0.0479	0.0117



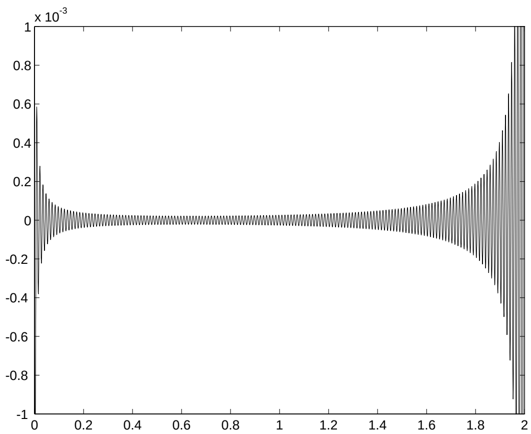


FIGURE 6.1. Example 6.1: $|(R_{321}-R_\infty)/R_\infty|$. $R(r)=e^{-|r|}$ is considered on $[-2,2]$. The graphs depict $|(R_{321}-R_\infty)/R_\infty|$ on $[-4,4]$ (upper) and $[0,2]$ (lower). Only the first 401 terms in the expansion $R_\infty(r)=\sum\rho_i^2\cos(\omega_i r)$ were taken ($\sum_{i=1}^{401}\rho_i^2\sim 0.9995$).

$\frac{1}{\pi}\frac{1}{1+\omega^2}\,d\omega$ is in Σ (with $T=2$). Fix n and let $\delta=T/(n-1)$ and $\bar\sigma_\delta \overset{d}{=} \sigma\circ\varphi_\delta^{-1}$. We next explicitly compute the infinite Jacobi matrix J associated with $\bar\sigma_\delta$. Since $\bar\sigma_\delta\in\bar\Sigma_n$, the $n\times n$ leading submatrix of J is the Jacobi matrix of $\bar\sigma_n$. For $\gamma\in[-2/\delta,2/\delta]$, let $\omega(\gamma)\overset{d}{=}\varphi_\delta^{-1}(\gamma)=2/\delta\arcsin(\gamma\delta/2)$. Then,

$$d\bar\sigma_\delta(\gamma)=\sum_{\omega:\varphi_\delta(\omega)=\gamma} d\sigma(\omega)\frac{d\omega}{d\gamma}d\gamma=\frac{d\gamma}{\sqrt{1-\left(\frac{\delta}{2}\gamma\right)^2}}\sum_{k\in\mathbb{Z}}\frac{1}{\pi}\frac{1}{1+\left(\omega(\gamma)+k\frac{2\pi}{\delta}\right)^2}$$
$$=\frac{d\gamma}{\sqrt{1-\left(\frac{\delta}{2}\gamma\right)^2}}\,\frac{\delta}{4}\,\frac{\sinh(\delta)}{\sinh^2(\frac{\delta}{2})+\sin^2(\omega(\gamma)\frac{\delta}{2})}.$$

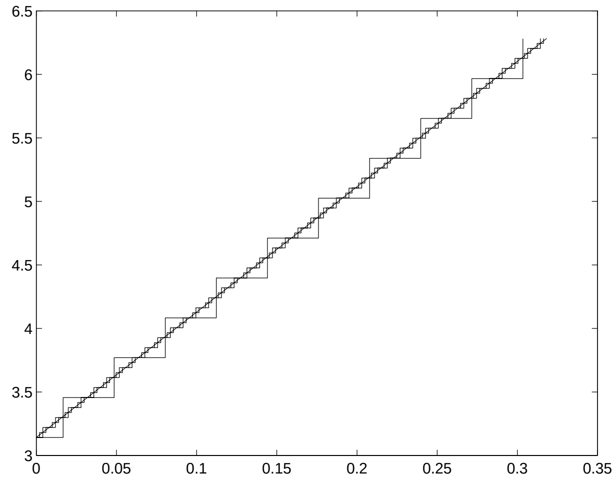


FIGURE 6.2. Example 6.1: Convergence of $m_{\bar{\sigma}_{2n+1}}$ to m_c. Again, $R(r) = e^{-|r|}$ is considered on $[-2,2]$. The straight line is the mass function $m_c(x) = \pi + \pi^2 x$ of the common string S_c. The stairs are the mass functions $m_{\bar{\sigma}_{2n+1}}$ of the strings corresponding to $\bar{\sigma}_{2n+1}$ for $n = 10, 40, 160$.

Thus, with $\nu(x) \overset{d}{=} \bar{\sigma}_\delta(2x/\delta)$ and $\mu = -\sinh^{-2}(\delta/2)$, for $-1 \le x \le 1$,

$$d\nu(x) = \frac{1}{\pi} \frac{dx}{\sqrt{1-x^2}} \frac{\sqrt{1-\mu}}{1-\mu x^2}.$$

This is a weight function studied by Grinšpun [4, section VI.13], and the coefficients in the three-term recursion formula of the orthogonal polynomials (the entries in the associated Jacobi matrix J) are: $a_i = 0$ for all i (J has vanishing diagonal) and

$$\tilde{b}_0 = 1, \qquad \tilde{b}_1 = \sqrt{\frac{1-e^{-\delta}}{2}}, \qquad \tilde{b}_2 = \frac{\sqrt{1+e^{-\delta}}}{2}, \qquad \tilde{b}_i = \frac{1}{2}, \quad i \ge 3.$$

The recursion coefficients of $\bar{\sigma}_\delta$ are now given by a simple scaling argument: $a_i = 0$ for all i, $b_0 = 1$, and for $i \ge 1$, $b_i = 2/\delta \tilde{b}_i$.

With the Jacobi matrices explicitly computed, finding σ_n is a rather trivial numerical task. Table 6.1 describes the convergence of σ_{2n} and σ_{2n+1} to σ_0 and σ_∞, respectively (see Example 3.5 as to how these two measures can be computed). Figure 6.1 illustrates the relative error between R_{321} and R_∞ (corresponding to the spectral functions σ_{321} and σ_∞, respectively) on a larger interval, $[-4,4]$. Finally, Figure 6.2 describes the convergence of the mass functions $m_{\bar{\sigma}_{2n+1}}$ of the strings corresponding to $\bar{\sigma}_{2n+1}$, to m_c, the mass function of the string S_c from Claim 3.3. m_c was provided in Example 3.5, and $m_{\bar{\sigma}_{2n+1}}$ was computed from its moments, $\{(-1)^k \Delta^{2k} R(0) : k = 0, \ldots, 2n\}$, as explained in Section 4. The strings of σ_{2n+1} are harder to compute.

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