# A POSSIBLE DEFINITION OF A STATIONARY TANGENT

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ABSTRACT. This paper offers a way to construct a locally optimal stationary approximation for a non-stationary Gaussian process. In cases where this construction leads to a unique stationary approximation we call it a stationary tangent. This is the case with Gaussian processes governed by smooth $n$-dimensional correlations. We associate these correlations with equivalence classes of curves in $\mathbb{R}^n$. These are described in terms of "curvatures" (closely related to the classical curvature functions); they are constant if and only if the correlation is stationary. Thus, the stationary tangent, at $t = t_0$, to a smooth correlation, curve or process, is the one with the same curvatures at $t_0$ (but constant). We show that the curvatures measure the quality of a local stationary approximation and that the tangent is optimal in this regard. These results extend to the smooth infinite-dimensional case although, since the equivalence between correlations and curvatures breaks down in the infinite-dimensional setting, we cannot, in general, single out a unique tangent. The question of existence and uniqueness of a stationary process with given curvatures is intimately related with the classical moment problem and is studied here by using tools from operator theory. In particular, we find that there always exists an optimal Gaussian approximation (defined via the curvatures). Finally, by way of discretizing we introduce the notion of $\delta$-curvatures designed to address non-smooth correlations.

## 1. Introduction

Stationary processes have been thoroughly studied and a comprehensive mathematical theory has been developed based primarily on the spectral distribution function. As a natural extension people considered "locally stationary processes". Intuitively, these are *non*-stationary processes that on a sufficiently small time scale do not deviate considerably from stationarity. To mention just a couple of approaches:

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Priestley has studied this problem through what he defines as the *evolutionary spectrum* [13, p. 148] and later he [14], and independently Mallat, Papanicolaou & Zhang [11], looked at the problem from wavelets point of view. These authors base their work on an analysis of the (real) correlation function. Such common practice can be justified, for example, by considering zero-mean real Gaussian processes which are studied in this paper as well.

The related question that we consider here was suggested to us by McKean: can one define a "stationary tangent" for a non-stationary Gaussian process? The linear tangent, at $t_0$, to a function $f$, is the best local linear approximation to $f$ at $t_0$. One can compute it from $f'(t_0)$, provided the function is differentiable. Similarly, the stationary tangent defined in this paper is an optimal stationary approximation to a sufficiently smooth non-stationary, say, correlation. Consider the following example. The correlation $R(t,s) = \cos\big(\omega(t) - \omega(s)\big)$, is stationary if and only if $\omega$ is a linear function of $t$, but for any differentiable $\omega$ and for $t, s$ near $t_0$:

$$R(t,s) = \cos\big(\dot\omega(t_0)(t-s)\big) + O(\rho^3),$$

where $\rho = \sqrt{(t-t_0)^2 + (s-t_0)^2}$. Thus, we are tempted to declare that,

$$\tilde R_{t_0}(t,s) = \cos\big(\dot\omega(t_0)(t-s)\big)$$

is the stationary tangent to $R$ at $t_0$, and it is indeed so according to our definition.

We say that the stationary process $\hat{\mathbb{X}}$ is an optimal stationary approximation to the process $\mathbb{X}$ at $t_0$, if it sequentially minimizes $\mathrm{E}\left|\mathbb{X}^{(k)}_{t_0} - \hat{\mathbb{X}}^{(k)}_{t_0}\right|^2$ for $k = 0, 1, 2 \ldots$ More precisely, consider the following decreasing sets of processes:

$$A_0 \overset{d}{=} \left\{\hat{\mathbb{X}} : \hat{\mathbb{X}} \text{ is stationary and } \mathrm{E}\left|\mathbb{X}_{t_0} - \hat{\mathbb{X}}_{t_0}\right|^2 = 0\right\}$$

$$A_n \overset{d}{=} \left\{\hat{\mathbb{X}} \in A_{n-1} : \mathrm{E}\left|\mathbb{X}^{(n)}_{t_0} - \hat{\mathbb{X}}^{(n)}_{t_0}\right|^2 = \min_{\mathbb{Y}\in A_{n-1}} \mathrm{E}\left|\mathbb{X}^{(n)}_{t_0} - \mathbb{Y}^{(n)}_{t_0}\right|^2\right\}$$

Then, $\hat{\mathbb{X}}$ is an optimal stationary approximation if $\hat{\mathbb{X}} \in \cap_n A_n$. We show that $\tilde{\mathbb{X}}$, our stationary tangent to $\mathbb{X}$ at $t_0$, is such an optimal approximation (Theorem 4). In some cases it is the unique such optimal approximation.

At this point we can ask what is the order of the optimal stationary approximation, or what is the maximal $d$ for which a.s. $\left|\mathbb{X}_{t_0} - \hat{\mathbb{X}}_{t_0}\right| = O\left(t-t_0\right)^{d+1}$. We call this maximal $d$, the order of stationarity of $\mathbb{X}$ at

$t_0$. Note that if $\mathrm{E}\left|\mathbb{X}_{t_0}-\hat{\mathbb{X}}_{t_0}\right|^2=0$, then a.s. $\left|\mathbb{X}_{t_0}-\hat{\mathbb{X}}_{t_0}\right|=O\left(t-t_0\right)$. Thus, the optimal stationary approximation, which we defined above, is indeed also optimal in the a.s. sense. We show that the same mechanism that allows us to define the stationary tangent can also be used to the determine the order of stationarity. We next provide a rough outline of this mechanism.

Throughout this paper we consider three related (smooth) objects: a 0-mean real Gaussian process, $\mathbb{X}$, a correlation function, $R$, and a curve $\boldsymbol{x}$ (in $\mathbb{R}^n$ or $l^2$), which we associate with $R$ via $R(t,s)=\langle \boldsymbol{x}_t, \boldsymbol{x}_s\rangle$. The main idea is to associate "curvatures" to our equivalence classes of curves, and thereby to the corresponding correlations and processes. These curvatures are positive functions closely related to the classical curvature functions and, as explained next, are well suited for the problem of stationary approximations. Indeed, stationary objects have constant curvatures which yields, in principle, a way to define a stationary tangent. For example, the stationary tangent correlation at $t_0$ to the correlation $R$, is the one which its *constant* curvatures are equal to the curvatures of $R$ at $t_0$. For processes, the definition of the tangent is slightly more involved, but in either cases we show it is an optimal stationary approximation.

In describing the tangent correlation we implicitly assumed that the curvatures of $R$ at $t_0$ uniquely determine a stationary correlation. However, this is only assured to be the case if the associated curve is in $\mathbb{R}^n$, a case which is studied in section 2 of this paper. More generally, while the curvatures approach will always yield an optimal stationary approximation, it might not yield a unique such approximation. As we show, the question of reconstructing a stationary tangent from its constant curvatures is equivalent to Stieltjes' moment problem (section 3.3 explains more about the connection with the moment problem).

Using results from the theory of evolution equation in a Hilbert space and of self-adjoint extensions of symmetric operators, we study in section 3.2 the general case of constant curvatures. In particular, we settle the aforementioned question of existence and uniqueness of an optimal stationary approximation, which in the case of non-uniqueness, we call a curvature stationary approximation. Note that an "$n$-dimensional" correlation (i.e., the associated curve is in $\mathbb{R}^n$) is stationary if and only if its curvatures are constant. However, for general curves in $l^2$, the correlation might not be stationary even though its curvatures are constant (Claim 3.8).

Finally, since the curvatures of any stationary process $\mathbb{X}$ have vanishing derivatives, the aforementioned order of stationarity depends on, roughly, how many derivatives of the curvature functions of $\mathbb{X}$ vanish at $t_0$. Thus, we can readily determine the order of stationarity of $\mathbb{X}$ from the derivatives of the curvatures. Naturally, the tangent, being optimal, is a stationary approximation of this order. Similarly, we define and treat the order of stationarity of a correlation and a curve.

In a following paper we will show how the curvature scheme can be extended to non-smooth correlations, as well as to non-instantaneous stationary approximations. The basic idea, as presented here in section 4, is to use finite differences instead of derivatives.

This paper contains research I did toward my Ph.D. title under the supervision of Henry McKean at the Courant Institute. I would like to take this opportunity to thank Henry for the numerous discussions we had over this subject and others.

## 2. Stationary tangents in the finite-dimensional setting

### 2.1. **Correlation, curves and curvatures.**

Any correlation $R$ of a mean-square continuous Gaussian process defined on a compact time interval $I$, can be expressed, by Mercer's theorem, as

$$R(t,s) = \sum_{i=1}^{\infty} \lambda_i e_i(t) e_i(s),$$

where $\lambda_i > 0$ are the eigenvalues and $e_i$'s are the eigenvectors of the integral operator defined by the kernel $R$. Thus, with $x_i(t) \stackrel{d}{=} \sqrt{\lambda_i} e_i(t)$,

$$R(t,s) = \sum_{i=1}^{\infty} x_i(t) x_i(s). \tag{1}$$

At first we will only consider correlations for which the sum in (1) is finite.

**Definition:**

- A correlation $R$ is $n$-dimensional if the sum in (1) extends up to $n$ and if the $x_i$'s are linearly independent.
- A Gaussian process is $n$-dimensional if its correlation is such.

We define an equivalence relation on (continuous) curves $\boldsymbol{x}_t \stackrel{d}{=} [x_1(t), x_2(t), \ldots, x_n(t)] \in \mathbb{R}^n$ as follows: The curves $\boldsymbol{x}$ and $\boldsymbol{y}$ are considered equivalent, if there exists a fixed orthogonal transformation of $\mathbb{R}^n$, $U$ such that $\boldsymbol{y} = U\boldsymbol{x}$. Let $[\boldsymbol{x}]$ denote the equivalence class of $\boldsymbol{x}$. Then we can associate an $n$ dimensional correlation with $[\boldsymbol{x}]$ via $R(t,s) \stackrel{d}{=} \langle \boldsymbol{x}_t, \boldsymbol{x}_s \rangle$, where $\langle , \rangle$ is the standard inner-product in $\mathbb{R}^n$. Conversely, given $R$, (1) yields a corresponding curve, and we find

*Claim 2.1.* There is a 1:1 onto correspondence between continuous correlations of dimension $\leq n$, and equivalence classes of (continuous) curves in $\mathbb{R}^n$.

*Proof.* Since orthogonal transformations preserve the inner-product, for any $\boldsymbol{y} \in [\boldsymbol{x}]$, $R(t,s) = \langle \boldsymbol{y}_t, \boldsymbol{y}_s \rangle$, hence the correspondence is well defined on our equivalence classes and from (1) we learn it is onto. As for 1:1, suppose that for all $t, s \in I$, $\langle \boldsymbol{x}_t, \boldsymbol{x}_s \rangle = \langle \boldsymbol{y}_t, \boldsymbol{y}_s \rangle$. In particular it follows that if $\boldsymbol{x}_t = \boldsymbol{x}_s$ then $\boldsymbol{y}_t = \boldsymbol{y}_s$. Hence $U\boldsymbol{x}_t \mapsto \boldsymbol{y}_t$ is a well defined map between the traces of the two curves. Furthermore, $U$ can be extended uniquely to an orthogonal map between the subspaces generated by the traces of the curves. In particular it can be extended as an orthogonal map of $\mathbb{R}^n$, and hence $\boldsymbol{y} \in [\boldsymbol{x}]$. □

In this geometric context, stationary correlations have a distinctive property; since $R(0) = \langle \boldsymbol{x}_t, \boldsymbol{x}_t \rangle$ the curve obviously lies on a sphere in $\mathbb{R}^n$, and for $t, t+r \in I$,

$$\left\langle \frac{\boldsymbol{x}_t}{|\boldsymbol{x}_t|}, \frac{\boldsymbol{x}_{t+r}}{|\boldsymbol{x}_{t+r}|} \right\rangle = \frac{R(r)}{R(0)},$$

is independent of $t$. Hence the curves that are associated with stationary correlations are angle-preserving, or (borrowing Krein's terminology [9]) helical curves on a sphere in $\mathbb{R}^n$.

Next we introduce a variant of the classical curvature functions, but in order to do so we need to restrict attention to:

**Definition 2.2.** A curve $\boldsymbol{x} \in \mathbb{R}^N$ is s.n.d, or strongly $n$ dimensional ($N \geq n$), if it is $n$ times continuously differentiable, and for each $t \in I$, $\{\boldsymbol{x}_t^{(k)}\}_{k=0}^{n-1}$ are linearly independent, while $\{\boldsymbol{x}_t^{(k)}\}_{k=0}^{n}$ are dependent.

*Remark.* If $[\boldsymbol{x}] = [\boldsymbol{y}]$, then clearly $\boldsymbol{x}$ is s.n.d if and only if $\boldsymbol{y}$ is. Therefore we can talk about s.n.d correlations as well. Later, definition 2.10 will specify this in terms of $R$ itself.

Let $\boldsymbol{x} \in \mathbb{R}^n$ be s.n.d, and let $\{\boldsymbol{v}_i(t)\}_{i=0}^{n-1}$ be the result of the Gram-Schmidt procedure applied to $\{\boldsymbol{x}_t^{(i)}\}_{i=0}^{n-1}$ (normalized so that $\langle \boldsymbol{x}^{(i)}, \boldsymbol{v}_i \rangle > 0$).

**Definition 2.3:**

- The $i$-th *curvature function* of $\boldsymbol{x}$ is $\kappa_i \stackrel{d}{=} \langle \dot{\boldsymbol{v}}_{i-1}, \boldsymbol{v}_i \rangle$
- The *orthogonal frame* of $\boldsymbol{x}$ at time $t$ is the $n \times n$ matrix $V_t$ whose rows are $\boldsymbol{v}_i(t)$.
- The *curvature matrix* of $\boldsymbol{x}$ is the $n \times n$ skew-symmetric tridiagonal matrix valued function $K_t$ with $K_t(i, i+1) = \kappa_i(t)$ for $i = 1, \ldots, n-1$.

*Remark.* The classical curvature functions can be defined in an analogous way; to get those, apply the Gram-Schmidt procedure to $\{\boldsymbol{x}_t^{(i)} : i = \underline{1}, \dots, \underline{n}\}$ (see e.g. [16]). The reason for introducing our variant of the curvatures, is that the classical curvatures are invariant under Euclidean transformation of $\mathbb{R}^n$, while our equivalence classes of curves are determined up to orthogonal transformations.

The dynamics of the orthogonal frame (called the Frenet frame in the classical version) is described by:

*Claim 2.4.* For $i = 1, 2, \dots, n-1$, $\kappa_i(t) > 0$ and

$$\dot{V} = KV \tag{2}$$

*Proof.* Since the $\boldsymbol{v}_i$'s are orthonormal it follows that $\langle \dot{\boldsymbol{v}}_i, \boldsymbol{v}_j \rangle = -\langle \boldsymbol{v}_i, \dot{\boldsymbol{v}}_j \rangle$, hence (2) holds with a skew-symmetric matrix $K$. Since

$$\mathrm{Span}\{\boldsymbol{x}, \boldsymbol{x}^{(1)}, \boldsymbol{x}^{(2)}, \dots, \boldsymbol{x}^{(k-1)}\} = \mathrm{Span}\{\boldsymbol{v}_0, \boldsymbol{v}_1, \dots, \boldsymbol{v}_{k-1}\},$$

it is obvious that $\langle \dot{\boldsymbol{v}}_i, \boldsymbol{v}_j \rangle = 0$ for $j \geq i+2$, hence $K$ is tridiagonal. Finally by our definition of the Gram-Schmidt process, $\langle \boldsymbol{x}^{(i)}, \boldsymbol{v}_i \rangle > 0$ hence $\kappa_i = \langle \dot{\boldsymbol{v}}_{i-1}, \boldsymbol{v}_i \rangle > 0$. □

We show next that with one more curvature function, $\kappa_0 \overset{d}{=} |\boldsymbol{x}|$, the curvatures characterize the equivalence classes of s.n.d curves, and therefore s.n.d. correlations.

Note that if we define $\kappa_k \overset{d}{=} 0$ for $k \notin \{0, \dots, n-1\}$, then (2) can be rewritten as

$$\dot{\boldsymbol{v}}_k = -\kappa_k \boldsymbol{v}_{k-1} + \kappa_{k+1} \boldsymbol{v}_{k+1} \qquad k = 0, \dots, n-1. \tag{3}$$

Repeatedly differentiating the equation $\boldsymbol{x} = \kappa_0 \boldsymbol{v}_0$ while using (3) yields:

$$\begin{aligned} \boldsymbol{x} &= \kappa_0 \boldsymbol{v}_0 \\ \dot{\boldsymbol{x}} &= \dot{\kappa}_0 \boldsymbol{v}_0 + \kappa_0 \kappa_1 \boldsymbol{v}_1 \\ \ddot{\boldsymbol{x}} &= \left(\ddot{\kappa}_0 - \kappa_0 \kappa_1^2\right) \boldsymbol{v}_0 + \left(2\dot{\kappa}_0 \kappa_1 + \kappa_0 \dot{\kappa}_1\right) \boldsymbol{v}_1 + \kappa_0 \kappa_1 \kappa_2 \boldsymbol{v}_2 \\ &\ \vdots \end{aligned} \tag{4}$$

More generally,

*Claim 2.5.* With $\boldsymbol{x}^{(k)} = \sum_{i=0}^{k} c_i^k \boldsymbol{v}_i$ (by induction on $k$) we have:

- $c_i^k$ are polynomials in $\{\kappa_l^{(j)} : 0 \leq l \leq k\,,\ 0 \leq j \leq k-l\}$.
- $c_k^k = \kappa_0 \kappa_1 \dots \kappa_k$, and $\kappa_k$ does not appear in $c_i^k$ for $i < k$.
- If $k \geq n$ this representation still holds (i.e., the $c_i^k$ are the same polynomials as for $k < n$) subject to $\kappa_{n+i} \equiv 0$ for all $i \geq 0$. In this case, for $i \geq n$, $c_i^k = 0$ and $\boldsymbol{v}_i$ is not defined.

*Claim 2.6.* Let $\boldsymbol{x}$ be an s.n.d curve. It defines $n$ positive curvature functions $\{\kappa_i(t)\}_{i=0}^{n-1}$, such that, $\kappa_i$ is $n-i$ times continuously differentiable, and these curvatures are shared by any curve which is equivalent to $\boldsymbol{x}$.

*Proof.* As stated in claim 2.5, $\langle \boldsymbol{x}^{(i)}, \boldsymbol{v}_i \rangle = \kappa_0\kappa_1 \dots \kappa_i$. Note that the left hand side is the length of the co-projection of $\boldsymbol{x}^{(i)}$ on the subspace generated by $\mathrm{Span}\langle \boldsymbol{x}, \boldsymbol{x}^{(1)}, \boldsymbol{x}^{(2)}, \dots, \boldsymbol{x}^{(i-1)} \rangle$, and hence with $\Omega_i$, the volume of the parallelepiped generated by $\{\boldsymbol{x}, \boldsymbol{x}^{(1)}, \boldsymbol{x}^{(2)}, \dots, \boldsymbol{x}^{(i)}\}$, we have:

$$\kappa_0\kappa_1 \dots \kappa_i = \frac{\Omega_i}{\Omega_{i-1}},$$

and hence

$$\kappa_i = \frac{\Omega_i \Omega_{i-2}}{\Omega_{i-1}^2},$$

where $\Omega_{-1} = \Omega_{-2} = 1$. It follows that if two curves belong to the same equivalence class, they define the same curvatures.

Note that $\Omega_i^2$ is the determinant of the Grammian matrix defined by $\boldsymbol{x}, \boldsymbol{x}^{(1)}, \boldsymbol{x}^{(2)}, \dots \boldsymbol{x}^{(i)}$:

$$D_i \stackrel{d}{=} \begin{vmatrix} \langle \boldsymbol{x}, \boldsymbol{x} \rangle & \langle \boldsymbol{x}, \boldsymbol{x}^{(1)} \rangle & \langle \boldsymbol{x}, \boldsymbol{x}^{(2)} \rangle & \dots & \langle \boldsymbol{x}, \boldsymbol{x}^{(i)} \rangle \\ \langle \boldsymbol{x}^{(1)}, \boldsymbol{x} \rangle & \langle \boldsymbol{x}^{(1)}, \boldsymbol{x}^{(1)} \rangle & \langle \boldsymbol{x}^{(1)}, \boldsymbol{x}^{(2)} \rangle & \dots & \langle \boldsymbol{x}^{(1)}, \boldsymbol{x}^{(i)} \rangle \\ \langle \boldsymbol{x}^{(2)}, \boldsymbol{x} \rangle & \langle \boldsymbol{x}^{(2)}, \boldsymbol{x}^{(1)} \rangle & \langle \boldsymbol{x}^{(2)}, \boldsymbol{x}^{(2)} \rangle & \dots & \langle \boldsymbol{x}^{(2)}, \boldsymbol{x}^{(i)} \rangle \\ \dots & & & & \\ \langle \boldsymbol{x}^{(i)}, \boldsymbol{x} \rangle & \langle \boldsymbol{x}^{(i)}, \boldsymbol{x}^{(1)} \rangle & \langle \boldsymbol{x}^{(i)}, \boldsymbol{x}^{(2)} \rangle & \dots & \langle \boldsymbol{x}^{(i)}, \boldsymbol{x}^{(i)} \rangle \end{vmatrix}, \tag{5}$$

hence

$$\kappa_i(t) = \frac{\sqrt{D_i D_{i-2}}}{D_{i-1}}, \tag{6}$$

where $D_{-1} = D_{-2} = 1$. Thus, as long as $\{\boldsymbol{x}, \boldsymbol{x}^{(1)}, \boldsymbol{x}^{(2)}, \dots, \boldsymbol{x}^{(i-1)}\}$ are linearly independent $D_{i-1} \neq 0$ and $\kappa_i(t)$ is well defined and indeed $n-i$ times continuously differentiable. Note that if $\{\boldsymbol{x}, \boldsymbol{x}^{(1)}, \dots, \boldsymbol{x}^{(i)}\}$ are linearly dependent, then by (6) $\kappa_i = 0$, which agrees with our previous definition. □

Suppose now that you are given the curvatures, or rather:

**Definition 2.7.** A curvature *type* matrix is a tridiagonal skew-symmetric matrix $K$ with $K(i, i+1) > 0$.

*Claim 2.8.* Let $U$ be an orthogonal matrix and let $K_t$ be a curvature type matrix valued function such that $\kappa_i \stackrel{d}{=} K_t(i, i+1)$ is $n-i$ times continuously differentiable. Then, there exists a unique s.n.d curve $\boldsymbol{x} \in \mathbb{R}^n$ such that, $K_t$ is the curvature matrix of $\boldsymbol{x}$ at time $t$ and $U$ is its orthogonal frame at $t = 0$.

*Proof.* Let $V_t$ be the unique solution of (2) with $V_0 = U$, and define $\boldsymbol{x}_t = \kappa_0(t)V_t^*\boldsymbol{e}_1$. Then, $\boldsymbol{x}$ is an $n$ times continuously differentiable curve in $\mathbb{R}^n$ (here we use the corresponding differentiability of the curvatures). Note that $V_t$ is an orthogonal matrix since $K_t$ is skew-symmetric. It is not hard to see that the result of Gram-Schmidt applied to $\{\boldsymbol{x}_t^{(k)}\}_{k=0}^{n-1}$ is $\{\boldsymbol{v}_i(t)\}_{i=0}^{n-1}$, where $\{\boldsymbol{v}_i(t)\}$ are the rows of $V_t$. It follows that $\boldsymbol{x}$ is s.n.d and that $V$ is its orthogonal frame. Since $\dot{V} = KV$, necessarily $K$ is the curvature matrix of $\boldsymbol{x}$. Finally, if $\boldsymbol{y}$ is another such s.n.d curve then, with $W_t$ being its orthogonal frame, $\dot{W} = KW$ and $W_0 = U$. Therefore, by the uniqueness of $V$, $W \equiv V$ and $\boldsymbol{y} \equiv \boldsymbol{x}$. □

*Claim 2.9.* Let $\{\kappa_i\}_{i=0}^{n-1}$ be $n$ positive (curvature) functions with $\kappa_i$ being $n - i$ times continuously differentiable. There exists a unique (up to equivalence) s.n.d curve, $\boldsymbol{x} \in \mathbb{R}^n$, with these curvatures.

*Proof.* Let $\boldsymbol{x}$ and $\boldsymbol{y}$ be two curves which share these curvatures. According to the previous claim, these are uniquely defined given their frames at $t = 0$, $V_{\boldsymbol{x}}(0)$, respectively, $V_{\boldsymbol{y}}(0)$. Let $U \stackrel{d}{=} V_{\boldsymbol{y}}(0)^{-1}V_{\boldsymbol{x}}(0)$ and let $\boldsymbol{w} \stackrel{d}{=} U\boldsymbol{x}$. Since the curvatures of $\boldsymbol{w}$ and $\boldsymbol{x}$ are identical, and since the frame of $\boldsymbol{w}$ is given by $V_{\boldsymbol{w}}(t) = V_{\boldsymbol{x}}(t)U^*$, it follows from the uniqueness part of the previous claim that $\boldsymbol{w} \equiv \boldsymbol{y}$. That is, $\boldsymbol{x}$ and $\boldsymbol{y}$ are in the same equivalence class. □

Since $R(t,s) = \langle \boldsymbol{x}_t, \boldsymbol{x}_s \rangle$, it follows that

$$D_i(t) = \begin{vmatrix} R & \partial_s R & \partial_s^2 R & \dots & \partial_s^i R \\ \partial_t R & \partial_t\partial_s R & \partial_t\partial_s^2 R & \dots & \partial_t\partial_s^i R \\ \partial_t^2 R & \partial_t^2\partial_s R & \partial_t^2\partial_s^2 R & \dots & \partial_t^2\partial_s^i R \\ \dots & \dots & \dots & \dots & \dots \\ \partial_t^i R & \partial_t^i\partial_s R & \partial_t^i\partial_s^2 R & \dots & \partial_t^i\partial_s^i R \end{vmatrix}, \tag{7}$$

where all the derivatives are evaluated at $(t,t)$. Using the above identity in (6) yields the curvatures in terms of $R$ directly. Furthermore definition 2.2 can now be phrased in terms of $R$.

**Definition 2.10.** A correlation $R$ is s.n.d if its derivatives $\partial_t^{(i)}\partial_s^{(j)}R$, $\max\{i,j\} \le n$, exist (and are continuous), and if for each $t \in I$, $D_{n-1}(t) > 0$, while $D_n \equiv 0$.

*Remark.* Suppose $R(t,s) = \langle \boldsymbol{x}_t, \boldsymbol{x}_s \rangle$. One can verify using (7) that $R$ is s.n.d if and only if $\boldsymbol{x}$ is such. Since we give a similar proof for the infinite-dimensional case in claim 3.1, we omit it here.

The following claim is now an immediate consequence.

*Claim 2.11.* There is a 1:1 correspondence between:

- s.n.d correlations
- equivalence classes of s.n.d curves
- $n$ positive curvature functions $\kappa_0, \dots \kappa_{n-1}$, where $\kappa_i$ is in $C^{n-i}(I)$.

*Remarks 2.12:*

- If $\boldsymbol{x}$ is an s.n.d curve in $\mathbb{R}^N$ with $N > n$, then by identifying the $n$-dimensional subspace spanned by $\{\boldsymbol{x}_t : t \in I\}$ with $\mathbb{R}^n$, we can define the $(n \times n)$ curvature matrix and the $(n \times N)$ orthogonal frame. The latter matrix will have orthonormal rows but it will obviously not be an orthogonal matrix.
- Suppose that $\boldsymbol{x} \in \mathbb{R}^n$ is a smooth curve such that $\{\boldsymbol{x}_t^{(i)}\}_{i=0}^{n-1}$ is linearly independent for all $t \in I$ except for a finite number of points. Then one can readily define the curvatures of this curve. The problem is that now there might be non-equivalent curves with the same curvatures. Consider for example the 1-dimensional curves $\boldsymbol{x}_t = (t^3)$ and $\tilde{\boldsymbol{x}}_t = (|t|^3)$. These are non-equivalent curves, yet their curvatures, $\kappa_0(t) = \tilde{\kappa}_0(t) = |t|^3$ and $\kappa_1 \equiv \tilde{\kappa}_1 \equiv 0$ are identical. The fact that $\tilde{\boldsymbol{x}}$ is not entirely smooth at $t = 0$ is irrelevant to the phenomenon; there is indeed a discontinuity in the orthogonal frame, but it is not reflected in the curve itself.

2.2. **Curvatures and stationary correlations.** The next theorem is the motivation behind the introduction of the curvatures.

**Theorem 1.** *The curvatures are constant if and only if the s.n.d correlation is stationary.*

*Remark.* In particular any $n$ dimensional stationary correlation is an s.n.d one. In view of that we will be omitting the acronym s.n.d where there is no room for confusion.

*Proof.* It follows immediately from (6) and (7) that if the correlation is stationary, then the curvatures are constant. On the other hand, if the curvatures are constant (2) is explicitly solvable: $V_t = e^{tK}V_0$. We can assume $V_0 = I$ and since $\boldsymbol{x}_t = \kappa_0(t)V_t^*\boldsymbol{e}_1$ always holds, we get

$$\boldsymbol{x}_t = \kappa_0 e^{-tK}\boldsymbol{e}_1.$$

In particular

$$R(t,s) = \langle \kappa_0 e^{-tK}\boldsymbol{e}_1, \kappa_0 e^{-sK}\boldsymbol{e}_1\rangle = \kappa_0^2\langle \boldsymbol{e}_1, e^{(t-s)K}\boldsymbol{e}_1\rangle,$$

which is obviously a stationary correlation. □

In proving the last theorem we found that, in the stationary case, $R(r) = \kappa_0^2\langle e^{rK}\boldsymbol{e}_1, \boldsymbol{e}_1\rangle$. Using the spectral resolution for the skew-symmetric $K$, $K = \sum_j i\omega_j \boldsymbol{u}_j \otimes \boldsymbol{u}_j$ (where $\pm i\omega_j$ are the eigenvalues and $\boldsymbol{u}_j$ are the eigenvectors), we find:

$$R(r) = \kappa_0^2 \Big\langle \exp\big(\sum_j ir\omega_j \boldsymbol{u}_j \otimes \boldsymbol{u}_j\big)\boldsymbol{e}_1, \boldsymbol{e}_1 \Big\rangle \tag{8}$$
$$= \kappa_0^2 \sum_j e^{ir\omega_j} |\boldsymbol{u}_j(1)|^2.$$

This identifies the spectral distribution function of $R$ as the spectral measure of the Hermitian matrix $iK$ obtained from the generating vector $\boldsymbol{e}_1$. In fact it is the same as the spectral measure of the real, symmetric matrix, $\hat{K}$, obtained from $K$ by flipping the signs of the elements in the lower sub-diagonal:

**Definition 2.13.** Let $A$ be a real skew-symmetric tridiagonal matrix. The *symmetrization* of $A$, is a symmetric tridiagonal matrix, $\hat{A}$ that is identical to $A$ but with $\hat{A}(j+1,j) = -A(j+1,j)$.

*Claim 2.14.* If $A$ is a real skew-symmetric tridiagonal matrix, then $iA$ and $\hat{A}$ are unitary equivalent matrices. Moreover, their spectral measures with respect to $\boldsymbol{e}_1$ are identical.

*Proof.* Let $U$ be the diagonal unitary matrix with $U(j,j) = i^j$, then $U(iA)U^{-1} = \hat{A}$. In other words, $iA$ is the same as $\hat{A}$ represented by the basis $\{i^j \boldsymbol{e}_j\}$. Since $\boldsymbol{e}_1$ is an eigenvector of $U$, the corresponding spectral measures are identical. □

Going back to (8), we notice that since $K$ is real, the eigenvalues $i\omega_j$ come in conjugate pairs, as do the corresponding eigenvectors $\boldsymbol{u}_j$. Thus, with $m = [\frac{n+1}{2}]$,

$$R(r) = \kappa_0^2 \sum_{j=1}^{m} \rho_j^2 \cos(\omega_j r),$$

where $\rho_j^2 = 2|\boldsymbol{u}_j(1)|^2$, except if $n$ is odd, in which case $\omega_1 = 0$ is an eigenvalue and $\rho_1^2 = |\boldsymbol{u}_1(1)|^2$. This shows that, assuming smoothness, a finite dimensional stationary correlation is necessarily of the type just mentioned (i.e., the spectral distribution function is discrete with $n+1$ jumps). One can show that is also the case without assuming smoothness, which is then a corollary (see [12]). In terms of the paths, we have the following geometric interpretation: any angle preserving (helical) curve on a finite dimensional sphere is equivalent to a bunch of circular motions, of radii $\rho_i$ and angular velocities $\omega_i$, performed in orthogonal planes.

Finally a word of caution. So far we discussed finite dimensional processes defined on a finite time interval. In the stationary case, if one wants to talk about the spectral distribution function, then tacitly it is assumed that there is one and only one way to extend the given stationary correlation to the whole line. In [8] Krein shows you can always extend a positive definite function (or a correlation of a stationary process) to the whole line, retaining its positive definite character. This extension might be unique, as is the case for the finite dimensional correlations we are dealing with here.

2.3. **Tangents.** Theorem 1 allows us to introduce various notions of tangent, all based on the idea of freezing the curvatures. There are three objects which are of interest here: the correlation, the curve, and the process itself. For each of these we will define its stationary tangent and then justify the terminology.

**Definition.** Let $R$ be an s.n.d correlation, its *stationary tangent* at $(t_0, t_0)$, is the correlation defined by the (constant) curvatures $\kappa_0(t_0), \dots, \kappa_{n-1}(t_0)$.

*Remarks:*

- If $R$ is stationary, then its tangent at $t_0$ is itself.
- Let $R$ be an s.n.d correlation, then given all its tangents $\tilde{R}_{t_0}$, $t_0 \in I$, we can reconstruct $R$ on $I \times I$ (since we know the curvatures at any point).

**Definition.** Let $\boldsymbol{x}$ be an s.n.d curve in $\mathbb{R}^n$ and let $K$ be its curvature matrix, and $V$ its orthogonal frame. The *stationary tangent curve* to $\boldsymbol{x}$ at $t_0$ is the stationary curve $\tilde{\boldsymbol{x}}$ defined by:

- The curvature matrix of $\tilde{\boldsymbol{x}}$ is $\tilde{K} \equiv K_{t_0}$
- The orthogonal frame of $\tilde{\boldsymbol{x}}$ at $t = t_0$ is $V_{t_0}$.

*Remarks:*

- By claim 2.8, there exists a unique tangent curve, furthermore,

$$\tilde{\boldsymbol{x}}_t = \kappa_0(t_0) \left[ \exp\big((t - t_0) K_{t_0}\big) V_{t_0} \right]^* \boldsymbol{e}_1,$$

- The definition of the tangent curve and the last equation hold also for an s.n.d curve $\boldsymbol{x} \in \mathbb{R}^N$ with $N > n$ (cf. remarks 2.12).
- If $\boldsymbol{x}$ is stationary, then $\tilde{\boldsymbol{x}} \equiv \boldsymbol{x}$ (claim 2.8 again).
- If $\tilde{\boldsymbol{x}}$ is the tangent curve to $\boldsymbol{x}$ at $t_0$, then $\tilde{R}(t,s) = \langle \tilde{\boldsymbol{x}}_t, \tilde{\boldsymbol{x}}_s \rangle$ is the tangent correlation to $R$ at $(t_0, t_0)$.

Finally, we define the tangent process. Let $\mathbb{X}$ be an s.n.d Gaussian process, i.e., it has an s.n.d correlation $R$. Then, there exists an $n$

dimensional Gaussian vector, $\boldsymbol{\xi} = [\xi_1, \xi_2, \ldots, \xi_n]$ (where $\xi_i$ are independent $N(0,1)$ random variables), and an s.n.d curve, $\boldsymbol{x}_t$, such that,

$$\mathbb{X}_t = \langle \boldsymbol{\xi}, \boldsymbol{x}_t \rangle = \sum_{i=1}^{n} \xi_i x_i(t). \tag{9}$$

*Remark.* The Karhunen-Loeve expansion is an example of such a representation: with $R(t,s) = \sum_{i=1}^{n} \lambda_i e_i(t) e_i(s)$ (spectral decomposition of $R$), one defines $\xi_i \stackrel{d}{=} \int_I \mathbb{X}_t e_i(t)\, dt$ ([10]), and we add $x_i(t) \stackrel{d}{=} \sqrt{\lambda_i} e_i(t)$.

**Definition 2.15.** The *stationary tangent process* at $t_0$ is $\tilde{\mathbb{X}}_t \stackrel{d}{=} \langle \boldsymbol{\xi}, \tilde{\boldsymbol{x}}_t \rangle$ where $\tilde{\boldsymbol{x}}$ is the stationary tangent curve to $\boldsymbol{x}$ at $t_0$.

*Remarks 2.16:*

- The representation $\mathbb{X} = \langle \boldsymbol{\xi}, \boldsymbol{x} \rangle$ where $\boldsymbol{\xi}_i$ are $N(0,1)$ independent random variables is not unique. However, if $\mathbb{X} = \langle \boldsymbol{\xi}, \boldsymbol{x} \rangle = \langle \boldsymbol{\eta}, \boldsymbol{y} \rangle$, where $\boldsymbol{\eta}_i$ are also independent and $N(0,1)$, then $R(t,s) = \langle \boldsymbol{x}_t, \boldsymbol{x}_s \rangle = \langle \boldsymbol{y}_t, \boldsymbol{y}_s \rangle$, so $\boldsymbol{x} = U\boldsymbol{y}$ for some fixed orthogonal $U$ and it follows that $\boldsymbol{\xi} = U\boldsymbol{\eta}$. Thus the tangent process is well-defined.
- $\tilde{\mathbb{X}}$ is a stationary process and it is jointly Gaussian with $\mathbb{X}$, meaning that any linear combination,
$$\sum_i \alpha_i \mathbb{X}_{t_i} + \sum_j \beta_j \tilde{\mathbb{X}}_{t_j} \qquad \alpha_i, \beta_j \in \mathbb{R},$$
is a Gaussian random variable.
- The correlation of $\tilde{\mathbb{X}}$ is the tangent correlation at $t_0$ (in particular, it is s.n.d).
- If $\mathbb{X}$ is stationary, then $\tilde{\mathbb{X}} \equiv \mathbb{X}$.
- Since the map $T : \mathbb{R}^n \to L^2(dP)$, described by $T\boldsymbol{x}_t \stackrel{d}{=} \mathbb{X}_t = \langle \boldsymbol{\xi}, \boldsymbol{x}_t \rangle$, is a well-defined isometry, Gram-Schmidting $\mathbb{X}_t^{(k)}$ is equivalent to Gram-Schmidting $\boldsymbol{x}_t^{(k)}$. Thus, for each $t$, there exist $n$ independent $N(0,1)$ random variables, $\mathbb{V}_k(t) \stackrel{d}{=} T(v_k(t))$, such that, with $c_i^k$ as in claim 2.5,
$$\mathbb{X}_t^{(k)} = \sum_{i=0}^{k} c_i^k \mathbb{V}_k(t).$$
This yields another way to describe the tangent process:
$$\tilde{\mathbb{X}}_t = \langle \boldsymbol{\xi}_t, \tilde{\boldsymbol{x}}_t \rangle = \kappa_0(t_0)[\mathbb{V}_0(t_0), \mathbb{V}_1(t_0), \ldots, \mathbb{V}_{n-1}(t_0)] \exp(-(t-t_0)K_{t_0})\boldsymbol{e}_1.$$
Note that to find $\mathbb{V}_k(t_0)$ from $\mathbb{X}_t^{(k)}$, we need "only" to compute $c_i^k$, which can be done directly from the correlation $R$.

So far we gave some justification for the usage of the term "stationary tangent". However, the most important one is that, as we will see in the next section, the tangent provide us with an optimal stationary approximation. Before we get there, let us take another look at the example from the introduction.

*Example 2.17.* The stationary tangent at $t_0$ to:

- the correlation $R(t,s) = \cos(\omega(t) - \omega(s))$ is,
$$\tilde{R}(t,s) = \cos(\dot{\omega}(t_0)(t-s)).$$
- the curve $\boldsymbol{x}_t = [\cos(\omega(t)), \sin(\omega(t))]$ is
$$\tilde{\boldsymbol{x}}_t = \Big[\cos\big(\omega(t_0) + \dot{\omega}(t_0)(t-t_0)\big), \sin\big(\omega(t_0) + \dot{\omega}(t_0)(t-t_0)\big)\Big].$$
- the process: $\mathbb{X}_t = \xi\cos(\omega t) + \eta\sin(\omega t)$, where $\xi$ and $\eta$ are independent $N(0,1)$, is
$$\tilde{\mathbb{X}}_t = \xi\cos\big(\omega(t_0) + \dot{\omega}(t_0)(t-t_0)\big) + \eta\sin\big(\omega(t_0) + \dot{\omega}(t_0)(t-t_0)\big).$$

This simple example agrees with our intuition. There are examples of non-trivial tangents, as we shall see later on.

2.4. **Curvatures and stationary approximations.** This section explains why the curvatures are well adapted for the study of local stationary approximations.

Let $\boldsymbol{x}$ and $\bar{\boldsymbol{x}}$ be strongly $n/\bar{n}$ dimensional curves in $\mathbb{R}^N$ with corresponding frames and curvatures, $\boldsymbol{v}_i/\bar{\boldsymbol{v}}_i$, respectively, $\kappa_i/\bar{\kappa}_i$. Then,

*Claim 2.18.*

$$\boldsymbol{x}_{t_0}^{(k)} = \bar{\boldsymbol{x}}_{t_0}^{(k)} \qquad \text{for } k = 0, \dots, m \tag{10}$$

if and only if

$$\kappa_k^{(p)}(t_0) = \bar{\kappa}_k^{(p)}(t_0) \qquad \text{and} \qquad \boldsymbol{v}_k(t_0) = \bar{\boldsymbol{v}}_k(t_0) \quad \text{for } k = 0, \dots, m\,,\ 0 \le p \le m-k \tag{11}$$

*Proof.* By claim 2.5, (10) follows immediately from (11). Conversely, if (10) holds, then obviously $\boldsymbol{v}_k(t_0) = \bar{\boldsymbol{v}}_k(t_0)$ for $k = 0, \dots, m$ (hence if $m \ge n$, necessarily $n = \bar{n}$). Let $N_k(t) \stackrel{d}{=} \langle \boldsymbol{x}_t^{(k)}, \boldsymbol{x}_t^{(k)} \rangle$. Then, (10) implies that $N_k^{(p)}(t_0) = \bar{N}_k^{(p)}(t_0)$ for $k + p \le m$. A somewhat technical argument, which is postponed to claim 5.2, now shows that $\kappa_k^{(p)}(t_0) = \bar{\kappa}_k^{(p)}(t_0)$ for $k + p \le m$. □

In view of the last claim and the fact that stationary curves have constant curvatures, we define

**Definition 2.19.** The *order of stationarity* at $t_0$ of an s.n.d curve, $\boldsymbol{x}$, is:

$$d(t_0) \stackrel{d}{=} \min\{m : \kappa_k^{(p)}(t_0) \neq 0 \text{ with } k \leq m-1, 1 \leq p \leq m-k\} - 1.$$

Note that if $\boldsymbol{x}$ is stationary, then $d \equiv \infty$. The following theorem is an immediate corollary of the last claim. Let $\boldsymbol{x}$ be an s.n.d curve with $d \stackrel{d}{=} d(t_0) < \infty$, then

**Theorem 2a.** *About $t = t_0$, $\hat{\boldsymbol{x}}$ is a stationary, local approximation to $\boldsymbol{x}$, of optimal order, if and only if*

$$\hat{\kappa}_i \equiv \kappa_i(t_0) \quad \textit{and} \quad \hat{v}_i(t_0) = v_i(t_0) \qquad \textit{for} \quad i = 0, \dots, d.$$

*In this case, $\|\hat{\boldsymbol{x}}_t - \boldsymbol{x}_t\| = O(|t-t_0|^{d+1})$ but $O(|t-t_0|^{d+2})$ fails. In particular, the tangent curve is a stationary approximation of optimal order.*

As for processes, let $\mathbb{X}$ and $\hat{\mathbb{X}}$ be strongly $n/\hat{n}$ dimensional Gaussian process which are jointly Gaussian. Then, $\mathbb{X}_t = \langle \boldsymbol{\xi}, \boldsymbol{x}_t \rangle$ and $\hat{\mathbb{X}}_t = \langle \boldsymbol{\xi}, \hat{\boldsymbol{x}}_t \rangle$ where $\boldsymbol{\xi}$ is a vector of independent $N(0,1)$ random variables and $\boldsymbol{x}$ and $\hat{\boldsymbol{x}}$ are s.n.d curves in $R^N$, where $N \geq \max(n, \hat{n})$. We can use

$$\mathrm{E}\left|\mathbb{X}_t - \hat{\mathbb{X}}_t\right|^2 = \|\boldsymbol{x}_t - \hat{\boldsymbol{x}}_t\|^2,$$

as a measurement of the quality of the approximation. An immediate corollary is that, about $t = t_0$, $\hat{\mathbb{X}} = \langle \boldsymbol{\xi}, \hat{\boldsymbol{x}} \rangle$ is a stationary local approximation of optimal order to the process $\mathbb{X}$, if and only if $\hat{\boldsymbol{x}}$ is an approximation of optimal order to $\boldsymbol{x}$ at $t = t_0$. Thus, theorem 2a has an exact analogue in terms of processes (with $\mathbb{V}_k$ from remarks 2.16 replacing $v_k$). However, since $\mathbb{X}$ and $\hat{\mathbb{X}}$ are jointly Gaussian,

$$\begin{aligned} &\mathrm{Prob}\left\{|\mathbb{X}_t - \hat{\mathbb{X}}_t| = O(|t-t_0|^{k+1})\right\} > \varepsilon \\ &\qquad\qquad \iff \qquad \mathrm{Var}\left(\mathbb{X}_{t_0}^{(j)} - \hat{\mathbb{X}}_{t_0}^{(j)}\right) = 0 \qquad j \leq k \\ &\qquad\qquad \iff \qquad \|\boldsymbol{x}_{t_0}^{(j)} - \hat{\boldsymbol{x}}_{t_0}^{(j)}\|^2 = 0 \qquad j \leq k. \end{aligned}$$

Thus, we get the bonus in terms of almost sure properties as stated in the following theorem. Let $d = d(t_0)$ be as in definition 2.19, then

**Theorem 2b.**
- *If at $t = t_0$, $\hat{\mathbb{X}}$ is a stationary approximation of optimal order, then*

  $$\hat{\kappa}_i \equiv \kappa_i(t_0) \qquad i = 0, \dots, d,$$

  *and*

  $$|\hat{\mathbb{X}}_t - \mathbb{X}_t| = O(|t-t_0|^{d+1}) \qquad a.s.$$

- *The tangent process is an optimal stationary approximation.*

- *For any stationary process, $\hat{\mathbb{X}}$, jointly Gaussian with $\mathbb{X}$,*

$$\text{Prob}\left\{|\mathbb{X}_t - \hat{\mathbb{X}}_t| = O(|t-t_0|^{d+2})\right\} = 0.$$

To prove the analogous result for correlations we need the following claim, the proof of which is postponed to section 5. In what follows, $R$ is an s.n.d correlation.

*Claim 2.20.* There is a perfect equivalence between $\{\partial^k R|_{(t_0,t_0)} : k \le n\}$ and $\{\kappa_k^{(p)}(t_0) \,:\, 0 \le k \le [n/2]\,,\, 0 \le p \le n-2k\}$.

**Definition 2.21.** The order of stationarity of $R$ at $(t_0, t_0)$ is defined as

$$d(t_0) \stackrel{d}{=} \min\{2k + p : p \ge 1, k \ge 0, \kappa_k^{(p)}(t_0) \ne 0\} - 1.$$

Let the stationary order of $R$ at $t_0$ be $d = d(t_0) < \infty$, then

**Theorem 3.** *About the point $(t_0, t_0)$, $\hat{R}$ is a stationary local approximation, of optimal order, to the correlation $R$, if and only if $\hat{\kappa}_i \equiv \kappa_i(t_0)$ for $i = 0, 1, \ldots, [d/2]$. In particular, the stationary tangent correlation is such. For any optimal order approximation, $|S(t,s) - R(t,s)| = O(\rho^{d+1})$ but $|S - R| \ne O(\rho^{d+2})$ (where $\rho = \sqrt{(t-t_0)^2 + (s-t_0)^2}$).*

*Proof.* Let $R$ and $\hat{R}$ be two smooth correlations. By claim 2.20, $|R - \hat{R}| = O(\rho^{n+1})$, about the point $(t_0, t_0)$, if and only if

$$\kappa_k^{(p)}(t_0) = \hat{\kappa}_k^{(p)}(t_0) \qquad \text{for } 0 \le k \le [n/2]\,,\, 0 \le p \le n-2k. \tag{12}$$

If $\hat{R}$ is stationary, then $\hat{\kappa}_k^{(p)} = 0$ for $p \ge 1$. Hence, (12) holds, if and only if $\kappa_k(t_0) = \hat{\kappa}_k(t_0)$ for $0 \le k \le [n/2]$, and $\kappa_k^{(p)} = 0$ for $0 \le k \le [n/2]$, $1 \le p \le n - 2k$. Since the latter can only hold if $n \le d$, it follows that $d+1$ is the optimal order for any stationary local approximation of $R$. Moreover, it is clear that the stationary tangent attains this optimal order. □

*Remark.* A variant of the previous theorems is obtained by assuming $\{\kappa_i(t) : i \le m-1\}$ are constant in an interval $I$. In this case, for any $t_0 \in I$, the stationary tangent at $t_0$ will yield $|R - S_{t_0}| = O(\rho^{2m+1})$ for correlations, and $O(|t-t_0|^m)$ for curves and processes. Furthermore, if $\kappa_m(t)$ is not constant in $I$, then this is the best we can do, in the sense that there will be points $t_0 \in I$ for which the next order of approximation fails.

At this point one might ask why is the tangent defined using more curvatures than the order of stationarity calls for? First note that you have to define the tangent with as many curvatures as the dimension requires, in order for your stationary tangent to be of the same dimension as the object you started with. Next, in order to be able to reconstruct the correlation, you need to know *all* the curvatures at any given point. The next statement applies only to curves and processes; their tangents are *unique* minimizers in a sense that is made precise as follows: Let $\tilde{\boldsymbol{x}}$ be the stationary tangent curve to $\boldsymbol{x}$ at $t_0$, and $\hat{\boldsymbol{x}}$ a stationary curve, then

**Theorem 4.**

$$\|\hat{\boldsymbol{x}}_{t_0}^{(j)} - \boldsymbol{x}_{t_0}^{(j)}\| = \|\tilde{\boldsymbol{x}}_{t_0}^{(j)} - \boldsymbol{x}_{t_0}^{(j)}\| \qquad 0 \le j \le k \tag{13}$$

*if and only if*

$$\hat{\kappa}_j = \kappa_j(t_0) \quad \textit{and} \quad \hat{\boldsymbol{v}}_j(t_0) = \boldsymbol{v}_j(t_0) \qquad 0 \le j \le k \tag{14}$$

*Furthermore, if* (13) *holds for* $k \ge -1$, *then*

$$\|\hat{\boldsymbol{x}}_{t_0}^{(k+1)} - \boldsymbol{x}_{t_0}^{(k+1)}\| \ge \|\tilde{\boldsymbol{x}}_{t_0}^{(k+1)} - \boldsymbol{x}_{t_0}^{(k+1)}\| \tag{15}$$

*Remark.* Replacing $\boldsymbol{v}_k$ with $\mathbb{V}_k$ (cf. remarks 2.16) and $\mathrm{E}\,|\mathbb{X}_t|^2$ with $\|\boldsymbol{x}_t\|^2$ etc., we get the analogous result for processes, which was formulated slightly differently in the introduction.

*Proof.* (13) follows trivially from (14). For the other implication we use induction on $k$. For $k = 0$, $\|\tilde{\boldsymbol{x}}_{t_0} - \boldsymbol{x}_{t_0}\| = 0$, so obviously (13) implies $\hat{\boldsymbol{x}}_{t_0} = \kappa_0(t_0)\boldsymbol{v}_{t_0}$ and (14) holds. Assume by way of induction that (14) holds for $k-1$. Then, by claim 2.5,

$$\begin{aligned}\|\hat{\boldsymbol{x}}_{t_0}^{(k)} - \boldsymbol{x}_{t_0}^{(k)}\|^2 = &\sum_{i=0}^{k-1} \left(\hat{c}_i^k - c_i^k\right)^2 \\ &+ \|\hat{\kappa}_0\hat{\kappa}_1 \ldots \hat{\kappa}_k \hat{\boldsymbol{v}}_k(t_0) - \kappa_0(t_0)\kappa_1(t_0)\ldots\kappa_k(t_0)\boldsymbol{v}_k(t_0)\|^2.\end{aligned}$$

For $i \le k-1$, $\hat{c}_i^k$ depend only on $\hat{\kappa}_0, \ldots, \hat{\kappa}_{k-1}$ which, by the inductive assumption, are the same as $\kappa_0(t_0), \ldots, \kappa_{k-1}(t_0)$. Therefore, (13) holds only if

$$\begin{aligned}&\|\hat{\kappa}_0\hat{\kappa}_1 \ldots \hat{\kappa}_k \hat{\boldsymbol{v}}_k(t_0) - \kappa_0(t_0)\kappa_1(t_0)\ldots\kappa_k(t_0)\boldsymbol{v}_k(t_0)\|^2 \\ &\qquad = \|\tilde{\kappa}_0\tilde{\kappa}_1 \ldots \tilde{\kappa}_k \tilde{\boldsymbol{v}}_k(t_0) - \kappa_0(t_0)\kappa_1(t_0)\ldots\kappa_k(t_0)\boldsymbol{v}_k(t_0)\|^2 = 0.\end{aligned}$$

Hence

$$\hat{\boldsymbol{v}}_k(t_0) = \boldsymbol{v}_k(t_0) \qquad \text{and} \qquad \hat{\kappa}_k = \kappa_k(t_0),$$

which completes the induction. As for (15), the same induction should work: just start it with $k = -1$, and note that since

$$\|\tilde{\kappa}_0\tilde{\kappa}_1 \ldots \tilde{\kappa}_k \tilde{\boldsymbol{v}}_k(t_0) - \kappa_0(t_0)\kappa_1(t_0)\ldots\kappa_k(t_0)\boldsymbol{v}_k(t_0)\|^2 = 0,$$

(15) follows. ∎

Let $\boldsymbol{x}$ be an s.n.d curve of stationary degree $d < \infty$ and let $\tilde{\boldsymbol{x}}$ be its tangent curve, at $t_0$. From the last theorem, and theorem 2a, we can deduce:

*Claim 2.22.* For any stationary curve $\hat{\boldsymbol{x}}$,

$$0 < \lim_{t \to t_0} \frac{\|\tilde{\boldsymbol{x}}_t - \boldsymbol{x}_t\|}{|t - t_0|^{d+1}} = \|\tilde{\boldsymbol{x}}_{t_0}^{(d+1)} - \boldsymbol{x}_{t_0}^{(d+1)}\| \leq \underline{\lim} \frac{\|\hat{\boldsymbol{x}}_t - \boldsymbol{x}_t\|}{|t - t_0|^{d+1}},$$

with equality if and only if, $\hat{\kappa}_i \equiv \kappa_i(t_0)$ and $\hat{\boldsymbol{v}}_i = \boldsymbol{v}_i(t_0)$ for $i = 0, \ldots, d, \underline{d+1}$. In this case the lim inf is actually a regular limit.

*Remarks:*

- Replacing $\mathrm{E}\,|\mathbb{X}_t|^2$ with $\|\boldsymbol{x}_t\|^2$ etc., we get the analogous result for processes.
- The analogue for correlations is false. For example, it is possible that with $\tilde{R}$ the tangent to a correlation $R$, of degree 1, there exists a stationary correlation $\hat{R}$, with

$$\infty > \overline{\lim_{(s,t) \to (t_0, t_0)}} \frac{|\tilde{R}(t,s) - R(t,s)|}{\rho^2} > \overline{\lim} \frac{|\hat{R} - R|}{\rho^2} > 0.$$

## 2.5. **Examples.**

*Example 2.23.*

$$R(t,s) = \hat{R}\big[\varphi(t) - \varphi(s)\big],$$

where $\hat{R}$ is a stationary correlation and $\varphi$ is a smooth real function. Note that example 2.17 is a special case of this one. The corresponding curves are following along stationary paths, where the timing is dictated by $\varphi(t)$. One can check that for these correlations, $\kappa_i(t) = \dot{\varphi}(t)\hat{\kappa}_i$ for $i \geq 1$ and that $\kappa_0 \equiv \hat{\kappa}_0$. Therefore,

$$\tilde{R}_{t_0}(t - s) = \hat{R}\big[\dot{\varphi}(t_0)(t - s)\big],$$

as we would expect.

*Example 2.24.*

$$R(t,s) = \frac{1}{2}\cos(t^2 - s^2) + \frac{1}{2}\cos(t^3 - s^3).$$

Intuitively, the stationary tangent should be:

$$R_{t_0}(t - s) \approx \frac{1}{2}\cos\big[2t_0(t - s)\big] + \frac{1}{2}\cos\big[3t_0^2(t - s)\big].$$

$R$ is a 4-dimensional correlation with $\kappa_0 \equiv 1$, and positive $\kappa_1, \kappa_2, \kappa_3$; the rest of the curvatures vanish. Therefore the curvature tangent is of the form:

(16)

$$\tilde{R}_{t_0}(t-s) = \rho_1(t_0)^2 \cos\big[\omega_1(t_0)(t-s)\big] \; + \; \rho_2(t_0)^2 \cos\big[\omega_2(t_0)(t-s)\big],$$

where $\omega_i(t_0)$ $(i = 1, 2)$ are the eigenvalues of the symmetrized curvature matrix $\hat{K}(t_0)$ (claim 2.14). Computing these explicitly we find:

$$\omega_1(t_0) = 3t_0^2 + O(t_0^{-6}) \qquad \omega_2(t_0) = 2t_0 + O(t_0^{-5}).$$

The weights $\rho_i$ are somewhat harder to get explicitly, so you are invited to inspect a numerical representation in figures 1 and 2. Note that there is a fast convergence to the tangent we anticipated intuitively. The reason the curvature tangent does not resemble our intuition for $t \leq 1.5$ is related to the fact that the change in the intuitive "stationary frequency" is big relative to the frequency itself. Also at $t = 0$ there is a violation of the "strong 4 dimensionality". Since $\kappa_1 = \frac{1}{2}\sqrt{18t^4 + 8t^2}$, the degree of stationarity of this correlation is $d = 1$.

*Example 2.25.*

$$R(t,s) = \frac{1}{2}\cos\big[\cos(t) - \cos(s)\big] + \frac{1}{2}\cos\big[\sin(t) - \sin(s)\big].$$

Here we find

$$\kappa_0 \equiv 1\,,\; \kappa_1 \equiv \frac{1}{\sqrt{2}}\,,\; \kappa_2 = \frac{1}{\sqrt{2}}\sqrt{4\cos^4 t - 4\cos^2 t + 3}\,,\; \kappa_3 > 0.$$

*Intuitively* we expect the stationary tangent to be

$$\tilde{R}_{t_0}(t-s) \approx \frac{1}{2}\cos\big[\sin(t_0)(t-s)\big] + \frac{1}{2}\cos\big[\cos(t_0)(t-s)\big].$$

The stationary correlation at the right hand side, $S$, matches $\kappa_0$, and $\kappa_1$, but fails with $\kappa_2(t_0)$. Thus

$$|R(t,s) - S(t-s)| = O(\rho^4)$$

The curvature tangent matches $\kappa_2(t_0)$ and so

$$|R(t,s) - \tilde{R}_{t_0}(t-s)| = O(\rho^5).$$

Note that $R$ is $\pi/2$ periodic in $t$ and $s$, and at the lattice points $(i\pi/2, j\pi/2)$, its (strong) dimension is not 4. The theory developed so far can be readily extended to this kind of isolated points, where the dimension suddenly drops. It should be pointed out, that in this example, the accuracy of the curvature tangent at these points is better than at the regular points. As for how this tangent correlation looks: it is again of the form (16). The frequencies and radii (which are found from the spectrum of $\hat{K}$) are depicted in figures 3 and 4.

Finally, the stationary degree of the corresponding curve/process is 2, while the naive approach will only match one derivative. The bottom half of figure 5 compares the standard deviation of the error these two stationary approximations generate. Both the tangent and the intuitive approximation are defined at $t_0 = \pi/8$. To demonstrate theorem 4, the upper half of figure 5 compares the graphs of $\sigma\left[\frac{1}{6}\mathbb{X}^{(3)}_{t_0} - \frac{1}{6}\tilde{\mathbb{X}}^{(3)}_{t_0}\right]$ and $\sigma\left[\frac{1}{6}\mathbb{X}^{(3)}_{t_0} - \frac{1}{6}\bar{\mathbb{X}}^{(3)}_{t_0}\right]$, where $\bar{\mathbb{X}}$ is a stationary process that agrees with the first three curvatures of $\mathbb{X}$ at $t_0$ but $\bar{\kappa}_3 = 1.25$ throughout. The graphs meet at the two points, $t_0$, where $\kappa_3(t_0) = 1.25$. It happens that $\kappa_2'(\pi/4) = 0$, and by theorem 2b, the degree of stationarity jumps to 3 at this point. This is reflected in $\sigma\left[\mathbb{X}^{(3)}_{\pi/4} - \tilde{\mathbb{X}}^{(3)}_{\pi/4}\right] = 0$, as you can check in figure 5.

## 3. The smooth infinite-dimensional case

3.1. **The curvatures and the problems.** Only part of the theory just developed for the $n$-dimensional case holds in the smooth infinite-dimensional case. As in section 2.1, we can associate infinite-dimensional, continuous correlations, defined on $I \times I$, with equivalence classes of continuous paths defined on $I$. The only difference is that the paths are in $l^2$ now, so the sum (1) is infinite; the equivalence is up to an isometry of the space. Claim 2.1 holds with essentially the same proof.

As in the finite-dimensional case, we are interested in a subclass of correlations and paths, to which we can assign curvatures.

**Definition:**

- A curve $\boldsymbol{x}$ is s.i.d if for any $\boldsymbol{\varphi} \in l^2$, $\langle \boldsymbol{x}_t, \boldsymbol{\varphi} \rangle$ is a $C^\infty$ (real) function and if, for any $t \in I$, $\{\boldsymbol{x}^{(k)}_t : k \in \mathbb{N}\}$ is a linearly independent set in $l^2$.
- A correlation $R$ is strongly infinite-dimensional (s.i.d) if it is infinitely differentiable, and for all $i \in \mathbb{N}$ and $t \in I$, $D_i(t) > 0$ (where $D_i$ is defined as in (7)).

The next claim assures us that these two objects should indeed share the same title.

*Claim 3.1.* Suppose $R(t,s) = \langle \boldsymbol{x}_t, \boldsymbol{x}_s \rangle$ for all $t, s \in I$. Then $R$ is s.i.d if and only if the curve $\boldsymbol{x}$ is such, and

$$\partial^i_t \partial^j_s R|_{(t,s)} = \langle \boldsymbol{x}^{(i)}_t, \boldsymbol{x}^{(j)}_s \rangle \tag{17}$$

*Proof.* If $\boldsymbol{x}$ is s.i.d, then $(\boldsymbol{x}_{t+h} - \boldsymbol{x}_t)/h \to \dot{\boldsymbol{x}}_t$, weakly in $l^2$, so $\partial_t R$ exists and satisfies (17). This can be repeated indefinitely in both $t$ and $s$ to verify that $R$ is smooth and that (17) holds. Since the derivatives $\{\boldsymbol{x}_t^{(k)}\}$ are linearly independent, $D_i(t) > 0$ by (5), and $R$ is indeed s.i.d.

Conversely, if $R$ is smooth, then for a fixed $t$ and any $s$,

$$\left\langle \frac{\boldsymbol{x}_{t+h} - \boldsymbol{x}_t}{h}, \boldsymbol{x}_s \right\rangle \underset{h\to 0}{\longrightarrow} \partial_t R|_{(t,s)},$$

and since

$$\left\| \frac{\boldsymbol{x}_{t+h} - \boldsymbol{x}_t}{h} \right\|^2 = \frac{R(t+h,t+h) - 2R(t+h,t) + R(t,t)}{h^2} < C < \infty,$$

$(\boldsymbol{x}_{t+h} - \boldsymbol{x}_t)/h$ converges weakly in $l^2$ as $h \to 0$. It follows that $\langle \boldsymbol{x}_t, \boldsymbol{\varphi} \rangle$ is differentiable for any $\boldsymbol{\varphi}$ and that $\partial_t R|_{(t,s)} = \langle \dot{\boldsymbol{x}}_t, \boldsymbol{x}_s \rangle$. Again, this argument can be repeated to yield the smoothness of $\boldsymbol{x}$. Finally, $\{\boldsymbol{x}_t^{(k)}\}$ is linearly independent by the definition of $R$ and (5). ☐

Given an s.i.d correlation or curve we define its (infinite number of) curvatures exactly as we did in section 2.1 for the s.n.d case. Namely, given an s.i.d correlation $R$, its curvatures can be computed from (6) and (7). Alternatively, if $\boldsymbol{x}$ is an s.i.d curve, then let $\{\boldsymbol{v}_k(t) : k = 0, 1, 2, \dots\}$ be the result of the the Gram-Schmidt procedure applied to $\{\boldsymbol{x}_t^{(k)} : k = 0, 1, 2, \dots\}$ (normalized so that $\langle x_t^{(j)}, \boldsymbol{v}_j(t) \rangle > 0$), and define $\kappa_i$, $V_t$ and $K_t$ as the infinite-dimensional version of definition 2.3. Note that the orthogonal frame, $V_t$, has orthonormal rows, but is no longer necessarily an orthogonal matrix. Clearly, (3) holds, therefore the ode $\dot{V} = KV$ is well defined and valid. Claim 2.5 also holds essentially unchanged. Note that, by (7) (or (5)), $\kappa_i$ is now infinitely differentiable, and that as in the s.n.d case, stationary correlations have constant curvatures.

While an s.<u>n</u>.d correlation can be recovered from its curvatures, this is no longer the case for an s.<u>i</u>.d correlation, not even in the stationary case. Therefore, in general, we cannot single out a curvature tangent, and we are led to the introduction of a slightly weaker notion.

**Definition 3.2:**

- A correlation, $\tilde{R}$, is a curvature stationary approximation (c.s.a) to an s.i.d correlation $R$, at $(t_0, t_0)$, if $\tilde{\kappa}_i \equiv \kappa_i(t_0)$.
- A curve, $\tilde{\boldsymbol{x}}$, is a c.s.a to an s.i.d curve $\boldsymbol{x}$, at $t_0$, if $\tilde{\kappa}_i \equiv \kappa_i(t_0)$ and $\tilde{\boldsymbol{v}}_i(t_0) = \boldsymbol{v}_i(t_0)$.

It is important to note that, as we show in section 3.2, an s.i.d curve $\boldsymbol{x} \in l^2$ always has a c.s.a, $\tilde{\boldsymbol{x}}$, at $t_0$ (corollary 5), and therefore an s.i.d correlation also has a c.s.a.

The results for curves and correlations from section 2.4 can now be trivially extended to the s.i.d case. The order of stationarity as defined in definitions 2.19 and 2.21 remains unchanged, as do claims 2.18 and 2.20. Replacing "stationary tangent" with "c.s.a" we obtain the s.i.d versions of theorems 2a, 3 and 4, and claim 2.22.

As for processes, recall that the representation (9), $\mathbb{X}_t = \langle \boldsymbol{\xi}, \boldsymbol{x}_t \rangle$, allowed us to essentially reduce the problem of the tangent process to that of the tangent curve. The same representation holds for s.i.d processes as well. Indeed, the Karhunen-Loeve theorem implies the existence of a vector, $\boldsymbol{\xi}$, of i.i.d $N(0,1)$ random variables, and of an s.i.d curve, $\boldsymbol{x}$, such that for each $t \in I$, $\mathrm{E}\left[\sum_1^n \xi_i x_i(t) - \mathbb{X}_t\right]^2 \to 0$. It follows by a result of Itô and Nisio [5, Theorem 4.1], that the series $\sum_1^n \xi_i x_i(t)$ converges a.s. to $\mathbb{X}$ uniformly on $I$. Moreover, since for any fixed $k$, the same holds for the series $\sum_1^n \xi_i x_i^{(k)}(t)$, as in the s.n.d case, $\mathbb{X}^{(k)} = \left\langle \boldsymbol{\xi}, \boldsymbol{x}^{(k)} \right\rangle$. Thus, definition 2.15 of the tangent process, theorem 2b, and the process variants of theorem 4 and claim 2.22 all hold for s.i.d processes as well, subject to the usual proviso that "stationary tangent" should be understood as c.s.a.

### 3.2. **Curves with constant curvatures.**

*Remark.* By considering the curve $\boldsymbol{x}/\kappa_0$ instead of $\boldsymbol{x}$, we can assume without loss of generality that $\kappa_0 \equiv 1$.

Let $K$ be the curvature matrix of an s.i.d $\boldsymbol{x}$. Let $\boldsymbol{u}_k(t)$ be the $k$-th column of the orthogonal frame of $\boldsymbol{x}$, $V_t$. Then,

$$\dot{\boldsymbol{u}}_k = K \boldsymbol{u}_k. \tag{18}$$

Thus, the question of existence and uniqueness of a curve with a given curvature matrix $K$, is related to the study of the evolution equation (18). We next explore this connection in the case of a constant curvature matrix.

Let $K$ be a curvature type matrix (def. 2.7) and, as in the finite-dimensional case, define $\hat{K}$ by flipping the sign of the entries in the lower sub-diagonal of $K$. As in the proof of claim 2.14, let $U$ be the unitary operator defined by $U\boldsymbol{e}_j = i^j \boldsymbol{e}_j$. Then, on the subspace $L \subset l^2$ of finitely supported sequences, $K = -U^{-1} i \hat{K} U$. $\hat{K}$ is a symmetric operator on $L$, therefore we can close it, and it is not hard to verify that its deficiency indices are either $(0,0)$ or $(1,1)$ [1, sec. 4.1.2]. We will say $\hat{K}$ (or $K$) is $(0,0)$ if these are the deficiency indices of $\hat{K}$. If $\hat{K}$ is $(0,0)$ then its closure is self-adjoint and it is the unique self-adjoint extension of $\hat{K}$. Else, $\hat{K}$ is $(1,1)$ and there is a one-parameter family of self-adjoint extensions of $\hat{K}$. These self-adjoint extensions of $\hat{K}$ are restrictions of its adjoint, $\hat{K}^*$, to a proper domain. It can be verified that $\hat{K}^*$ is the natural operator one would associate with a matrix, i.e., $(\hat{K}^*\boldsymbol{u})_i \stackrel{d}{=} \sum_j \hat{K}_{ij}\boldsymbol{u}_j$ and $\boldsymbol{u} \in \mathfrak{D}(\hat{K}^*)$ if and only if $\sum \left|(\hat{K}^*\boldsymbol{u})_i\right|^2 < \infty$. For more on this subject see e.g. [2, Ch. VII].

Let $\hat{A}$ be a self-adjoint extension of $\hat{K}$, and define

$$A \stackrel{d}{=} -U^{-1} i \hat{A} U.$$

Then it is easily verified that

$$\langle A\boldsymbol{e}_j, \boldsymbol{e}_k \rangle = \langle K\boldsymbol{e}_j, \boldsymbol{e}_k \rangle,$$

therefore $A$ is a skew-adjoint extension of $K$. Note that although $K$ was real, with respect to to the standard conjugation operator on $l^2$, $A$ is not necessarily so. Be that as it may, by Stone's theorem (e.g. [4, ch. 1]) $A$ is the generator of a $(C_0)$ unitary group of operators denoted by $e^{tA}$. That is, for any $\boldsymbol{\varphi} \in \mathfrak{D}(A)$, $\boldsymbol{u}_t \stackrel{d}{=} e^{tA}\boldsymbol{\varphi}$ satisfies $\dot{\boldsymbol{u}} = A\boldsymbol{u}$ and $\boldsymbol{u}_0 = \boldsymbol{\varphi}$. The group $e^{tA}$ will thus allow us to construct curves with a given curvature matrix. One should note though, that this general theory of operators is not entirely satisfactory from our point of view, since the ode (18) is well defined, coordinate wise, even if $\boldsymbol{u} \notin \mathfrak{D}(A)$.

We are only interested in real curves, denoted by $\boldsymbol{x}_t \in l^2(\mathbb{R})$, and since, in general, $A$ is not real, we need:

*Claim 3.3.* $e^{tA}\boldsymbol{\varphi} \in l^2(\mathbb{R})$ for any $t \in \mathbb{R}$ and $\boldsymbol{\varphi} \in l^2(\mathbb{R})$ if and only if $\rho$, the spectral measure of $\hat{A}$ with respect to $\boldsymbol{e}_1$, is symmetric i.e., $d\rho(x) = d\rho(-x)$.

*Proof.* Clearly, $e^{tA}\boldsymbol{\varphi} \in l^2(\mathbb{R})$ for all $\boldsymbol{\varphi}$ and $t$ if and only if $\langle e^{tA}\boldsymbol{e}_j, \boldsymbol{e}_k \rangle \in \mathbb{R}$ for all $t \in \mathbb{R}$ and $j, k \in \mathbb{N}$. Since

$$\frac{d^{n+m}}{dt^{n+m}} \langle e^{tA}\boldsymbol{e}_1, \boldsymbol{e}_1 \rangle = \langle e^{tA}A^n\boldsymbol{e}_1, (-1)^m A^m \boldsymbol{e}_1 \rangle = (-1)^m \langle e^{tA}K^n\boldsymbol{e}_1, K^m\boldsymbol{e}_1 \rangle,$$

and since $\mathrm{Span}\{\boldsymbol{e}_1, K\boldsymbol{e}_1, \dots, K^m\boldsymbol{e}_1\} = \mathrm{Span}\{\boldsymbol{e}_1, \dots, \boldsymbol{e}_{m+1}\}$, it follows that $e^{tA}\boldsymbol{\varphi} \in l^2(\mathbb{R})$ for all $t$ and $\boldsymbol{\varphi}$ if and only if $\langle e^{tA}\boldsymbol{e}_1, \boldsymbol{e}_1 \rangle \in \mathbb{R}$ for all $t$. Let $e^{-it\hat{A}}$ be the group of unitary operators generated by the skew-adjoint operator $-i\hat{A}$. It is not hard to verify that $e^{tA} = U^{-1}e^{-it\hat{A}}U$ and therefore

$$\langle e^{tA}\boldsymbol{e}_1, \boldsymbol{e}_1 \rangle = \langle U^{-1}e^{-it\hat{A}}U\boldsymbol{e}_1, \boldsymbol{e}_1 \rangle = \langle e^{-it\hat{A}}\boldsymbol{e}_1, \boldsymbol{e}_1 \rangle = \int e^{-it\lambda}\, d\rho(\lambda). \tag{19}$$

Hence $\langle e^{tA}\boldsymbol{e}_1, \boldsymbol{e}_1 \rangle \in \mathbb{R}$ for all $t$, if and only if

$$\int e^{-it\lambda}\, d\rho(\lambda) = \overline{\int e^{-it\lambda}\, d\rho(\lambda)} = \int e^{-it\lambda}\, d\rho(-\lambda),$$

for all $t$, which is equivalent to the symmetry of $\rho$. □

The next claim assures us that such symmetric spectral measures always exist.

*Claim 3.4.* (i) If $\hat{K}$ is essentially self-adjoint (the $(0,0)$ case), then its spectral measure with respect to $\boldsymbol{e}_1$ is symmetric.
(ii) If $\hat{K}$ is $(1,1)$, then it has exactly two self-adjoint extensions with a symmetric spectral measure with respect to $\boldsymbol{e}_1$.

*Proof.* The proof is a variation on theorem 2.13 in [15]. Let $W$ be the unitary operator defined on $l^2$ by $W\boldsymbol{e}_n = (-1)^{n+1}\boldsymbol{e}_n$. Then $W\hat{K}^*W^{-1} = -\hat{K}^*$. Let $\hat{A}$ be a self-adjoint extension of $\hat{K}$, so $\hat{K} \subset \hat{A} \subset \hat{K}^*$. We show next that $\mathfrak{D}(\hat{A})$ is an invariant subspace of $W$, or equivalently, that

$$W\hat{A}W^{-1} = -\hat{A}, \tag{20}$$

if and only if $d\rho_{\hat{A}}$, the spectral measure of the operator $\hat{A}$ with respect to $\boldsymbol{e}_1$, is symmetric. Indeed, if $\hat{A}$ satisfies (20) then

$$d\rho_{\hat{A}}(-\lambda) = d\rho_{-\hat{A}}(\lambda) = d\rho_{W\hat{A}W^{-1}}(\lambda) = d\rho_{\hat{A}}(\lambda).$$

Conversely, assume that $d\rho_{-\hat{A}}$ is symmetric and let $B \stackrel{d}{=} W\hat{A}W^{-1}$ and $C \stackrel{d}{=} -\hat{A}$. Then, $d\rho_B = d\rho_C$ and therefore

$$\langle e^{itB}\hat{K}^j\boldsymbol{e}_1, \hat{K}^k\boldsymbol{e}_1\rangle = \langle B^k e^{itB}B^j\boldsymbol{e}_1, \boldsymbol{e}_1\rangle = \langle C^k e^{itC}C^j\boldsymbol{e}_1, \boldsymbol{e}_1\rangle = \langle e^{itC}\hat{K}^j\boldsymbol{e}_1, \hat{K}^k\boldsymbol{e}_1\rangle.$$

This implies that $e^{itB} = e^{itC}$ for all $t$, and in particular their generators are identical so (20) holds.

If $\hat{K}$ is $(0,0)$, then $\hat{K}^*$ is the self-adjoint closure of $\hat{K}$ whence (20) holds, and (i) follows.

Suppose, on the other hand, that $\hat{K}$ is $(1,1)$. Von Neumann theory of extensions of symmetric operators (e.g. [2, Ch. VII]) guarantees the existence of normal eigenvectors $\boldsymbol{v}, \boldsymbol{w} \in l^2$ such that $\hat{K}^*\boldsymbol{v} = i\boldsymbol{v}$ and $\hat{K}^*\boldsymbol{w} = -i\boldsymbol{w}$. Specifying that $\langle \boldsymbol{v}, \boldsymbol{e}_1\rangle > 0$ and $\langle \boldsymbol{w}, \boldsymbol{e}_1\rangle > 0$ uniquely determines both. According to von Neumann, any self-adjoint extension of $\hat{K}$ can be uniquely characterized as follows. Choose $\theta \in [0, 2\pi)$, let $\boldsymbol{z}_\theta \stackrel{d}{=} (\boldsymbol{v} - e^{i\theta}\boldsymbol{w})/2i$, and let $\mathfrak{D}(\hat{A}_\theta) \stackrel{d}{=} \mathfrak{D}(\hat{K}) \oplus \mathrm{Span}\{\boldsymbol{z}_\theta\}$ where $\oplus$ stands here for a direct sum (not orthogonal). Then, $\hat{A}_\theta$, the restriction of $\hat{K}^*$ to $\mathfrak{D}(\hat{A}_\theta)$ is a distinct self-adjoint extension of $\hat{K}$. Using induction, one can verify that $W\boldsymbol{v} = \boldsymbol{w}$ and $W\boldsymbol{w} = \boldsymbol{v}$. It follows that for $\theta = 0$ and $\theta = \pi$, $\mathfrak{D}(\hat{A}_\theta)$ is an invariant subspace of $W$ and therefore $d\rho_{\hat{A}_\theta}$ is symmetric. Conversely, if $\theta \notin \{0, \pi\}$, $\mathrm{Span}\{\boldsymbol{z}_\theta, W\boldsymbol{z}_\theta\} = \mathrm{Span}\{\boldsymbol{v}, \boldsymbol{w}\}$, which implies that $\mathfrak{D}(\hat{A}_\theta)$ is not invariant under $W$, since by von Neumann,

$$\mathfrak{D}(\hat{K}^*) = \mathfrak{D}(\hat{K}) \oplus \mathrm{Span}\{\boldsymbol{v}\} \oplus \mathrm{Span}\{\boldsymbol{w}\}.$$

This completes the proof of (ii). □

Let $A$ be a skew-adjoint extension of $K$. As mentioned earlier, so far we only know that $e^{tA}\boldsymbol{\varphi}$ satisfies (18) if $\boldsymbol{\varphi} \in \mathfrak{D}(A)$. What if we start with $\boldsymbol{\varphi} \notin \mathfrak{D}(A)$?

*Claim 3.5.* For any $\boldsymbol{\varphi} \in l^2$, $\boldsymbol{u}_t \overset{d}{=} e^{tA}\boldsymbol{\varphi}$ satisfies

$$\frac{d}{dt}\langle \boldsymbol{u}_t, \boldsymbol{e}_j \rangle = \langle \boldsymbol{u}_t, -K\boldsymbol{e}_j \rangle \qquad \text{for } j \geq 1. \tag{21}$$

*Remark.* Note that the last equation is equivalent to (18) being satisfied coordinate wise.

*Proof.*

$$\begin{aligned} \frac{1}{h}\big(\langle \boldsymbol{u}_{t+h}, \boldsymbol{e}_j \rangle - \langle \boldsymbol{u}_t, \boldsymbol{e}_j \rangle\big) &= \frac{1}{h}\left\langle (e^{hA} - I)e^{tA}\boldsymbol{\varphi}, \boldsymbol{e}_j \right\rangle \\ &= \left\langle \boldsymbol{u}_t, \frac{1}{h}\left(e^{-hA} - I\right)\boldsymbol{e}_j \right\rangle. \end{aligned}$$

As $h \to 0$, the right hand side converges to $\langle \boldsymbol{u}_t, -A\boldsymbol{e}_j \rangle = \langle \boldsymbol{u}_t, -K\boldsymbol{e}_j \rangle$. □

We can now prove existence of a stationary curve with a given constant curvature matrix.

*Claim 3.6.* Let $K$ be a curvature type matrix and let $V_0$ be an orthogonal matrix. Let $A$ be a skew-adjoint extension of $K$ such that $d\rho_{\hat{A}}$ is symmetric, then the curve $\boldsymbol{x} \overset{d}{=} \left(e^{tA}V_0\right)^* \boldsymbol{e}_1$ satisfies:

(i) The constant curvature matrix of $\boldsymbol{x}$ is $K$.
(ii) The orthogonal frame of $\boldsymbol{x}$ at $t = 0$ is $V_0$.
(iii) $\boldsymbol{x}$ is a real stationary curve with correlation

$$R(t,s) = \int e^{i\lambda(t-s)}\, d\rho_{\hat{A}}(\lambda).$$

*Proof.* Let $V_t \overset{d}{=} e^{tA}V_0$ and denote its columns by $\boldsymbol{u}_k(t)$, $k = 1, 2, \ldots$, and its rows by $\boldsymbol{v}_n(t)$, $n \geq 0$. For each $k$, $\boldsymbol{u}_k(t) = e^{tA}\boldsymbol{u}_k(0)$, so by the previous claim, for $j \geq 1$

$$\frac{d}{dt}\langle \boldsymbol{u}_k, \boldsymbol{e}_j \rangle = \langle \boldsymbol{u}_k, -K\boldsymbol{e}_j \rangle = \kappa_j \langle \boldsymbol{u}_k, \boldsymbol{e}_{j+1} \rangle - \kappa_{j-1}\langle \boldsymbol{u}_k, \boldsymbol{e}_{j-1} \rangle,$$

where $\boldsymbol{e}_0 \overset{d}{=} 0$. Therefore with $\boldsymbol{v}_{-1} \overset{d}{=} 0$,

$$\dot{\boldsymbol{v}}_n = -\kappa_n \boldsymbol{v}_{n-1} + \kappa_{n+1}\boldsymbol{v}_{n+1}.$$

By definition, $\boldsymbol{x}_t = \boldsymbol{v}_0(t)$ and an elementary inductive argument shows that $\mathrm{Span}\{\boldsymbol{x}_t, \boldsymbol{x}_t^{(1)}, \dots, \boldsymbol{x}_t^{(n)}\} = \mathrm{Span}\{\boldsymbol{v}_0(t), \dots, \boldsymbol{v}_n(t)\}$, and that $\langle \boldsymbol{x}_t^{(n)}, \boldsymbol{v}_n(t)\rangle = \kappa_1, \dots \kappa_{n-1} > 0$. Since $V_t$ is an orthogonal matrix it follows that it is the orthogonal frame of $\boldsymbol{x}$ at time $t$ and that $K$ is the constant curvature matrix of $\boldsymbol{x}$. This proves (i) and (ii). Finally, by claim 3.3, $e^{tA}$ is real, thus $\boldsymbol{x}_t \in \mathbb{R}$ and

$$R(t,s) = \langle \boldsymbol{x}_t, \boldsymbol{x}_s\rangle = \langle V_0^{-1} e^{-tA}\boldsymbol{e}_1, V_0^{-1} e^{-sA}\boldsymbol{e}_1\rangle = \langle e^{(s-t)A}\boldsymbol{e}_1, \boldsymbol{e}_1\rangle,$$

which by (19) proves (iii). □

The last claim shows that a skew-adjoint extension of $K$ can yield a stationary curve satisfying (i) and (ii) of that claim. We show next that this is the only way to get such curves.

*Claim 3.7.* Let $\boldsymbol{x}$ be a (real) stationary curve. Let $K$ be the curvature matrix of $\boldsymbol{x}$ and let $V_t$ be its orthogonal frame at $t$. Suppose that $V_0$ is an orthogonal matrix, then there exists a skew-adjoint operator $A \supset K$ such that $\rho_{\hat{A}}$ is symmetric and $V_t = e^{tA}V_0$.

*Proof.* As usual, let $\boldsymbol{v}_i(t)$ be the orthonormal rows of $V_t$. One can show by induction, that for a stationary $\boldsymbol{x}$ (cf. claim 2.5),

$$\mathrm{Span}\{\boldsymbol{x}_t^{(2k)} : k = 0, \dots, n\} = \mathrm{Span}\{\boldsymbol{v}_{2k}(t) : k = 0, \dots, n\}$$
$$\mathrm{Span}\{\boldsymbol{x}_t^{(2k+1)} : k = 0, \dots, n\} = \mathrm{Span}\{\boldsymbol{v}_{2k+1}(t) : k = 0, \dots, n\}.$$

Therefore, $\langle \boldsymbol{x}^{(i)}(t), \boldsymbol{x}^{(j)}(s)\rangle = (-1)^{i-j}\langle \boldsymbol{x}^{(i)}(s), \boldsymbol{x}^{(j)}(t)\rangle$ implies that $|\langle \boldsymbol{v}_i(t), \boldsymbol{v}_j(s)\rangle| = |\langle \boldsymbol{v}_i(s), \boldsymbol{v}_j(t)\rangle|$. By assumption, $\{\boldsymbol{v}_i(0) : i \geq 0\}$ is an orthonormal basis, therefore for any $i$,

$$\sum_j |\langle \boldsymbol{v}_i(0), \boldsymbol{v}_j(t)\rangle|^2 = \sum_j |\langle \boldsymbol{v}_i(t), \boldsymbol{v}_j(0)\rangle|^2 = 1,$$

and we can conclude that $V_t$ is an orthogonal matrix for any $t$. Let $U_{ts}$ be the linear map defined by $U_{ts}\boldsymbol{u}_k(t) \overset{d}{=} \boldsymbol{u}_k(t+s)$, where $\boldsymbol{u}_k(t)$ are the columns of $V_t$. Since $V_t$ is an orthogonal matrix $U_{ts}$ is a well-defined orthogonal operator. Clearly $U_{ts} = V_{t+s}V_t^{-1}$, thus

$$\langle U_{ts}\boldsymbol{e}_k, \boldsymbol{e}_j\rangle = \langle V_t^*\boldsymbol{e}_k, V_{t+s}^*\boldsymbol{e}_j\rangle = \langle \boldsymbol{v}_{k-1}(t), \boldsymbol{v}_{j-1}(t+s)\rangle.$$

Since $\boldsymbol{x}$ is stationary, $\langle \boldsymbol{v}_{k-1}(t), \boldsymbol{v}_{j-1}(t+s)\rangle$ depends only on $s$, whence for all $t$, $U_{ts} = U_{0s}$ which we denote by $U_s$. It follows that $U_{t+s} = U_tU_s$ and by Stone's theorem $U_t = e^{tA}$ where $A$ is the skew-adjoint generator of the orthogonal group $U_t$. Since $\dot{V} = KV$, necessarily $A \supset K$. Finally, since $V_t = e^{tA}V_0$ is real and $V_0$ is orthogonal, it follows from claim 3.3 that $\rho_{\hat{A}}$ is symmetric. □

**Corollary 5.** *Let $W$ be a row-orthonormal matrix and let $K$ be a curvature type matrix. Then, $\hat{K}$ is either (0,0) or (1,1), according as there exist exactly one or two non-equivalent stationary curves $\boldsymbol{x}$ satisfying:*

(i) *The constant curvature matrix of $\boldsymbol{x}$ is $K$.*
(ii) *The orthogonal frame of $\boldsymbol{x}$ at $t = 0$ is $W$.*
(iii) $\overline{\operatorname{Span}\{\boldsymbol{x}_s : s \in \mathbb{R}\}} = \overline{\operatorname{Span}\{\boldsymbol{x}_0^{(k)} : k \in \mathbb{N}\}}$.

*Remark.* By the presumed stationarity of $R$, if (iii) holds at $t = 0$, it holds for all $t$. This condition is then equivalent to the fact that the Gaussian process governed by $R$ is completely predictable given all its derivatives at any given $t$.

*Proof.* By claim 3.4, $\hat{K}$ is either (0,0) or (1,1), according as there exist exactly one or two skew-adjoint extensions $A \supset K$ with a symmetric $\rho_{\hat{A}}$. For any such $A$, by claim 3.6, $\boldsymbol{y} \stackrel{d}{=} \left(e^{tA}\right)^* \boldsymbol{e}_1$ is a stationary curve with a constant curvature matrix $K$ and its orthogonal frame at 0 is $I$. Let $\boldsymbol{x} \stackrel{d}{=} W^*\boldsymbol{y}$. Since $W^*$ is an isometry, $\boldsymbol{x}$ is also a stationary curve with the curvature matrix $K$, and its orthogonal frame at 0 is $(W^*I)^* = W$. By the definition of $\boldsymbol{x}$, $\operatorname{Span}\{\boldsymbol{x}_t : t \in \mathbb{R}\} \subset \operatorname{Span}\{\boldsymbol{w}_i\}$, therefore $\boldsymbol{x}$ satisfies (i)-(iii). We next show that these are the only such curves: suppose that $\boldsymbol{x}$ is a curve that satisfies (i)-(iii). Then, $W$ considered as an operator on $\operatorname{Span}\{\boldsymbol{x}_t\}$ is an isometry. Thus, $\boldsymbol{y} \stackrel{d}{=} W\boldsymbol{x}$ is a stationary curve with the curvature matrix $K$ and its orthogonal frame at 0 is $I$. Therefore, by claim 3.7, $\boldsymbol{y}$ is one of the curves considered in the first part of the proof and so is $\boldsymbol{x}$. □

*Remark.* It is known that if $K$ is $(1, 1)$, then the spectral measure, $d\rho_{\hat{A}}$, of any self-adjoint extension $\hat{A} \supset \hat{K}$ is discrete (e.g. [15, theorem 5]). Thus, the curves, representing the corresponding correlations are, as in the finite-dimensional stationary case, just a bunch of circular motions. This should be contrasted with the curvatures that uniquely characterize $R(r) = \frac{\sin r}{r}$: there is no curve, representing $R$, that is composed of circular motions in orthogonal, 2-dimensional, planes.

The existence of two non-equivalent stationary curves with a (1,1) constant curvature matrix $K$ has a somewhat undesirable consequence:

*Claim 3.8.* If $\hat{K}$ is $(1, 1)$, then there exist *non*-stationary curves with the constant curvature matrix $K$.

*Proof.* Let $\hat{A}_1$ and $\hat{A}_2$ be the two non-equivalent extensions of $\hat{K}$ with symmetric $\rho_{\hat{A}_j}$. As usual, $A_j = -U^{-1}i\hat{A}_jU$ are the corresponding skew-adjoint extensions of $K$. Then, by claim 3.6, $\boldsymbol{x}_t \stackrel{d}{=} (e^{tA_1})^*\boldsymbol{e}_1$ and $\boldsymbol{y}_t \stackrel{d}{=}$

$(e^{tA_2})^* \boldsymbol{e}_1$ are two (real) stationary curves satisfying (i)-(iii) of that claim (with $V_0 = I$). Consider the curve

$$\boldsymbol{z}_t \stackrel{d}{=} \begin{cases} \boldsymbol{y}_t & t \leq 0 \\ \boldsymbol{x}_t & t > 0 \end{cases}.$$

Since the orthogonal frame of both $\boldsymbol{x}$ and $\boldsymbol{y}$, at $t = 0$, is $I$, and since their curvatures are identical, it follows that $\boldsymbol{z}$ is a smooth curve with the constant curvature matrix $K$. By (iii) of claim 3.6, $\boldsymbol{z}$ is non-stationary. Note that similarly we can construct any curve $\boldsymbol{z}$ that is made of segments along which $\boldsymbol{z}$ is alternately equivalent to $\boldsymbol{x}$ or to $\boldsymbol{y}$. □

In view of the last claim one might suspect that, even if $\hat{K}$ is $(0,0)$, there are non-stationary curves with the constant curvature matrix $K$. However, as we show next, this is not the case.

*Claim 3.9.* If the curvature type matrix $K$ is $(0,0)$, then for any $\boldsymbol{\varphi} \in l^2$ there exists a unique curve $\boldsymbol{u}$ in $l^2$ which is weakly continuous (i.e., $\boldsymbol{u}_t \to \boldsymbol{u}_{t_0}$ weakly in $l^2$, as $t \to t_0$), and such that $\boldsymbol{u}$ satisfies (21) and $\boldsymbol{u}_0 = \boldsymbol{\varphi}$.

*Remark.* Identifying $K$ with its skew-adjoint closure, we know that $\boldsymbol{u}_t \stackrel{d}{=} e^{tK}\boldsymbol{\varphi}$ is a solution of (21) with $\boldsymbol{u}_0 = \boldsymbol{\varphi}$. Furthermore, if we *assume* that $\boldsymbol{u}_t \in \mathfrak{D}(K)$ for all $t$, then by the "well-posedness theorem" (e.g. [4, sec. II.1.2]), $\boldsymbol{u}$ is the unique solution of the evolution equation $\dot{\boldsymbol{u}} = K\boldsymbol{u}$ , $\boldsymbol{u}_0 = \boldsymbol{\varphi}$. Thus, it is also the unique solution of (21).

*Proof.* Let $\boldsymbol{u}$ be such a coordinate wise smooth and weakly continuous solution. Define

$$\boldsymbol{w}_\delta(t) \stackrel{d}{=} \frac{1}{\delta} \int_t^{t+\delta} \boldsymbol{u}(s)\, ds.$$

Then, $\boldsymbol{w}_\delta(t) \in l^2$ and

$$\begin{aligned} \langle \boldsymbol{w}_\delta(t), -K\boldsymbol{e}_i \rangle &= \frac{1}{\delta} \int_t^{t+\delta} \langle \boldsymbol{u}(s), -K\boldsymbol{e}_i \rangle\, ds \\ &= \frac{1}{\delta} \int_t^{t+\delta} \frac{d}{ds} \langle \boldsymbol{u}(s), \boldsymbol{e}_i \rangle\, ds \\ &= \frac{1}{\delta} \left\langle \boldsymbol{u}(t+\delta) - \boldsymbol{u}(t), \boldsymbol{e}_i \right\rangle. \end{aligned}$$

Hence $\sum_i |\langle \boldsymbol{w}_\delta(t), -K\boldsymbol{e}_i \rangle|^2 < \infty$, and since $K$ is essentially skew-adjoint it follows that $\boldsymbol{w}_\delta(t) \in \mathfrak{D}(K)$ for all $t$. Since

$$\frac{d}{dt}\langle \boldsymbol{w}_\delta(t), \boldsymbol{e}_i \rangle = \frac{1}{\delta}\langle \boldsymbol{u}(t+\delta) - \boldsymbol{u}(t), \boldsymbol{e}_i \rangle = \langle \boldsymbol{w}_\delta(t), -K\boldsymbol{e}_i \rangle = \langle K\boldsymbol{w}_\delta(t), \boldsymbol{e}_i \rangle,$$

it follows that $\boldsymbol{w}_\delta$ satisfies $\dot{\boldsymbol{w}}_\delta = K\boldsymbol{w}_\delta$, and again by the "well-posedness theorem", $\boldsymbol{w}_\delta(t) = e^{tK}\boldsymbol{w}_\delta(0)$. Finally,

$$\langle \boldsymbol{w}_\delta(t), \boldsymbol{e}_i\rangle = \frac{1}{\delta}\int_t^{t+\delta} \langle \boldsymbol{u}(s), \boldsymbol{e}_i\rangle \, ds \xrightarrow[\delta\to 0]{} \langle \boldsymbol{u}(t), \boldsymbol{e}_i\rangle,$$

and

$$\|\boldsymbol{w}_\delta(t)\| \leq \frac{1}{\delta}\int_t^{t+\delta} \|\boldsymbol{u}(s)\| \, ds \leq C < \infty,$$

as we are considering a bounded time interval where $\|\boldsymbol{u}\|$ is bounded by the weak continuity assumption. Therefore, as $\delta \to 0$, $\boldsymbol{w}_\delta(t) \to \boldsymbol{u}(t)$ weakly in $l^2$. Thus,

$$\boldsymbol{u}(t) = \lim_{\delta\to 0} \boldsymbol{w}_\delta(t) = e^{tK} \lim_{\delta\to 0} \boldsymbol{w}_\delta(0) = e^{tK}\boldsymbol{\varphi},$$

where the limits are weak $l^2$. □

*Claim 3.10.* If $\hat{K}$ is $(0,0)$, then there exists a unique correlation with the constant curvature matrix $K$.

*Proof.* Let $\boldsymbol{x}$ be an s.i.d curve with the constant curvature matrix $K$, and let $V_t$ be its orthogonal frame at $t$. Note that we cannot assume $V_t$ is an orthogonal matrix. Let $\boldsymbol{u}_k(t)$ be the columns of $V_t$. Then, clearly, $\dot{\boldsymbol{u}}_k = K\boldsymbol{u}_k$. Since

$$\|\boldsymbol{u}_k(t)\|^2 = \sum_j \langle \boldsymbol{v}_j, \boldsymbol{e}_k\rangle^2 \leq \|\boldsymbol{e}_k\|^2 = 1,$$

and since as $t \longrightarrow t_0$,

$$\langle \boldsymbol{u}_k(t), \boldsymbol{e}_j\rangle = \langle \boldsymbol{v}_{j-1}(t), \boldsymbol{e}_k\rangle \longrightarrow \langle \boldsymbol{v}_{j-1}(t_0), \boldsymbol{e}_k\rangle = \langle \boldsymbol{u}_k(t_0), \boldsymbol{e}_j\rangle,$$

$\boldsymbol{u}_k$ is weakly continuous. Therefore by the last claim $V_t = e^{tK}V_0$, and it follows that

$$R(t,s) = \langle \boldsymbol{x}_t, \boldsymbol{x}_s\rangle = \langle V_0 V_0^* e^{-tK}\boldsymbol{e}_1, e^{-sK}\boldsymbol{e}_1\rangle = \langle e^{(s-t)K}\boldsymbol{e}_1, \boldsymbol{e}_1\rangle.$$

□

In view of the last claim, one might be tempted to guess that if $\hat{K}_t$ is (0,0) for all $t$, then there exists a unique correlation with the curvature matrices $K_t$. We do not know the answer to this question nor to the question whether any such curvature type matrix valued function is the curvature matrix of an s.i.d correlation. The next section mentions a different representation of these problems.

3.3. **Curvatures and orthogonal polynomials.** By now we know that the curvatures of a stationary correlation are intimately related with its derivatives, which in turn are the moments of the spectral distribution function (up to sign changes). Thus, we are naturally drawn into the classical moment problem: given a list of moments, is there a positive measure which produces them? In fact, the relation with the moment problem is more involved.

From the classical moment theory [1], we know that a sequence $\{M_k\}$ agrees with the moments of a positive measure, if and only if it is positive in the Hankel sense (i.e., $\sum_{i,k} M_{i+k}a_i a_k \geq 0$ for any $\boldsymbol{a} = (a_0, \ldots, a_{n-1}) \in \mathbb{R}^n$). Such sequences are in 1:1 correspondence with infinite symmetric, tridiagonal, Jacobi matrices (provided the moment sequence is normalized so that $M_0 = 1$). A convenient way to describe this correspondence is via the orthogonal polynomials. Indeed, if a positive measure $\Delta$ has moments of all order, then one can apply the Gram-Schmidt procedure to the powers $1, \gamma, \gamma^2, \ldots$, in the Hilbert space $L^2(\Delta)$, to get the orthonormal polynomials $\{P_k(\gamma)\}$ (note that the only information used in this process is the moments of the measure). In fact, one usually computes them using the famous three term recursion formula they satisfy [1]:

$$b_k P_{k+1}(\gamma) = (\gamma - a_k)P_k(\gamma) - b_{k-1}P_{k-1}(\gamma).$$

The coefficients in this recursion formula are the entries in the Jacobi matrix associated with the given list of moments. More precisely, the $\{a_n\}_{n\geq 0}$ are the entries on the main diagonal, while the strictly positive $\{b_n\}_{n\geq 0}$ are on the two sub-diagonals. Furthermore,

$$b_k = \frac{\sqrt{T_{k-1}T_{k+1}}}{T_k} \qquad \text{where} \tag{22}$$

$$T_k = \begin{vmatrix} M_0 & M_1 & M_2 & \ldots & M_k \\ M_1 & M_2 & M_3 & \ldots & M_{k+1} \\ M_2 & M_3 & M_4 & \ldots & M_{k+2} \\ \multicolumn{5}{c}{\dotfill} \\ M_k & M_{k+1} & M_{k+2} & \ldots & M_{2k} \end{vmatrix}. \tag{23}$$

It is not hard to verify that symmetric positive-definite sequences (i.e. with vanishing odd moments) are in 1:1 correspondence with Jacobi matrices with vanishing main diagonal.

The similarity to the curvature computation is not a coincidence:

*Claim 3.11.* For stationary correlations, $\kappa_k = b_{k-1}$.

*Proof.* In the stationary case, with

$$R(t,s) = S(t-s) \qquad \text{and} \qquad S(r) = \int e^{ir\gamma}\, d\Delta(\gamma),$$

$$\partial_t^j \partial_s^k R\Big|_{(t,t)} = (-1)^k S^{(j+k)}(0) = (-1)^k i^{j+k} M_{j+k}.$$

Hence, (7) can be written as,

$$D_k = \begin{vmatrix} M_0 & -iM_1 & -M_2 & iM_3 & \dots & (-i)^k M_k \\ iM_1 & M_2 & -iM_3 & -M_4 & \dots & (-i)^{k-1} M_{k+1} \\ -M_2 & iM_3 & M_4 & & \dots & (-i)^{k-2} M_{k+2} \\ -iM_3 & -M_4 & & & \dots & \\ & \dots & \dots & \dots & \dots & \\ i^k M_k & i^{k-1} M_{k+1} & i^{k-2} M_{k+2} & & \dots & M_{2k} \end{vmatrix}. \tag{24}$$

As in claim 2.14, the matrices that appear in (23) and (24) are unitary equivalent, in particular $D_k = T_k$. Comparing (22) with (6) completes the proof. □

*Remark 3.12.* The last claim shows the curvatures are a generalization of the Jacobi matrices to the non-stationary case, and that in the stationary case, the symmetrization of the curvature matrix, $\hat{K}$ (cf. definition 2.13), is exactly the Jacobi matrix.

The above discussion allows us to provide alternative proofs to some of the statements in section 3.2. For example, corollary 5 translates into the following statement about the classical moment problem: Let $K$ be a Jacobi matrix with vanishing diagonal. Then there exist exactly one or two positive measures $\sigma$ such that, $\sigma$ is symmetric, it solves the corresponding moment problem and the polynomials are dense in $L^2(\sigma)$, according as $K$ is (0,0) or (1,1). This statement can be deduced from [1], or it can be found, more explicitly, in [7, sec. 5.5].

The symmetric moment problem can be equivalently stated in terms of a real positive-definite function (alternatively, a stationary correlation), as follows: given a symmetric sequence $\{M_n\}$, does there exist a (unique) stationary correlation $R$ such that $R^{(k)}(0) = (-1)^k M_k$? In this case, as we know, the necessary condition which is the positivity of the determinants $D_i$ from (7) is also sufficient. The uniqueness can be settled in terms of the deficiency indices of the Jacobi matrix (cf. cor 5). This can be generalized as follows: given a sequence of smooth functions defined on the interval $I$, $\{M_{2n}(t)\}$, does there exist a (unique) correlation $R$, defined on $I \times I$, such that for any $t \in I$, $\partial_t^k \partial_s^k R|_{(t,t)} = M_{2k}(t)$? Note that (26) below shows that $\{\partial_t^k \partial_s^k R|_{(t_0,t_0)}\}$ determine all other derivatives of $R$ at $(t_0, t_0)$. As in the stationary case, an obvious necessary condition for the existence of such an $R$ is that $D_i > 0$, whether this is also sufficient and whether there is a similar criterion for uniqueness are the same unanswered questions that were raised at the end of the previous section.

## 4. A few words on $\delta$-curvatures

4.1. **The definition.** The previous sections provided us with a rather complete picture of the smooth case. This section offers a peek at our approach to the issues of non-smooth correlations as well as correlations that are given on finite time intervals. In this paper we restrict ourselves to a brief overview of the subject, a deeper study will appear in a following paper.

The basic idea is to replace derivatives with finite differences. Let $\delta > 0$ and $f$ be a function on $\mathbb{R}$, we define

$$\Delta f(t) \stackrel{d}{=} \frac{f(t+\delta/2) - f(t-\delta/2)}{\delta}.$$

Let $R$ be a continuous correlation on $[-T/2, T/2] \times [-T/2, T/2]$. With finite differences replacing derivatives, we define the $\delta$-curvatures of $R$ in an analogous way to (5)-(7). First, let

$$D_k \stackrel{d}{=} \begin{vmatrix} R & \Delta_s R & \Delta_s^2 R & \dots & \Delta_s^k R \\ \Delta_t R & \Delta_t \Delta_s R & \Delta_t \Delta_s^2 R & \dots & \Delta_t \Delta_s^k R \\ \Delta_t^2 R & \Delta_t^2 \Delta_s R & \Delta_t^2 \Delta_s^2 R & \dots & \Delta_t^2 \Delta_s^k R \\ \multicolumn{5}{c}{\dotfill} \\ \Delta_t^k R & \Delta_t^k \Delta_s R & \Delta_t^k \Delta_s^2 R & \dots & \Delta_t^k \Delta_s^k R \end{vmatrix},$$

where the finite differences are evaluated at $(0,0)$, and $k\delta \le T$. With $R(t,s) = \langle \boldsymbol{x}_t, \boldsymbol{x}_s \rangle$,

$$D_k = \begin{vmatrix} \langle \boldsymbol{x}, \boldsymbol{x} \rangle & \langle \boldsymbol{x}, \Delta \boldsymbol{x} \rangle & \langle \boldsymbol{x}, \Delta^2 \boldsymbol{x} \rangle & \dots & \langle \boldsymbol{x}, \Delta^k \boldsymbol{x} \rangle \\ \langle \Delta \boldsymbol{x}, \boldsymbol{x} \rangle & \langle \Delta \boldsymbol{x}, \Delta \boldsymbol{x} \rangle & \langle \Delta \boldsymbol{x}, \Delta^2 \boldsymbol{x} \rangle & \dots & \langle \Delta \boldsymbol{x}, \Delta^k \boldsymbol{x} \rangle \\ \langle \Delta^2 \boldsymbol{x}, \boldsymbol{x} \rangle & \langle \Delta^2 \boldsymbol{x}, \Delta \boldsymbol{x} \rangle & \langle \Delta^2 \boldsymbol{x}, \Delta^2 \boldsymbol{x} \rangle & \dots & \langle \Delta^2 \boldsymbol{x}, \Delta^k \boldsymbol{x} \rangle \\ \dots \\ \langle \Delta^k \boldsymbol{x}, \boldsymbol{x} \rangle & \langle \Delta^k \boldsymbol{x}, \Delta \boldsymbol{x} \rangle & \langle \Delta^k \boldsymbol{x}, \Delta^2 \boldsymbol{x} \rangle & \dots & \langle \Delta^k \boldsymbol{x}, \Delta^k \boldsymbol{x} \rangle \end{vmatrix},$$

where all differences are evaluated at 0. Hence as in the smooth case, $D_k$ is the square of the volume of the parallelepiped generated by $\boldsymbol{x}_0, \Delta \boldsymbol{x}_0, \dots, \Delta^k \boldsymbol{x}_0$. So with $D_{-1} \stackrel{d}{=} 1$, we can define the "$\delta$-curvatures" as

$$\kappa_k(\delta) \stackrel{d}{=} \frac{\sqrt{D_k D_{k-2}}}{D_{k-1}}, \tag{25}$$

provided $D_{k-1} > 0$ which is the analogue of the strong $n$/infinite-dimensional assumptions. If $R$ is $n$-dimensional, then, loosely speaking, for most $\delta$s, $D_{n-1}(\delta) > 0$.

*Remarks:*

- With the obvious modifications, this computation can be centered about any $t_0$.
- If $R$ is smooth, it is not hard to see that $\kappa_k(\delta) \xrightarrow[\delta\to 0]{} \kappa_k$.

One way to compute $\delta$-curvatures is as follows. Let $R(r) = \int_{\mathbb{R}} e^{ir\omega}\, d\sigma(\omega)$ be a stationary correlation, and define $S_\delta : [-\pi/\delta, \pi/\delta] \mapsto [-2/\delta, 2/\delta]$ to be the invertible map $S_\delta(\omega) \stackrel{d}{=} (2/\delta)\sin(\omega\delta/2)$. Define the symmetric positive measure $\bar{\sigma}_\delta$ on $[-2/\delta, 2/\delta]$ as $\bar{\sigma} \stackrel{d}{=} \sigma \circ S^{-1}$. Then, one can prove that

*Claim 4.1.* The *curvatures* of $\bar{R}_\delta = \int e^{ir\gamma}\, d\bar{\sigma}(\gamma)$ are the same as the $\delta$-curvatures of $R$.

*Remark.* Note that $\bar{\sigma}$ is supported on a compact set, thus $\bar{R}$ is indeed smooth.

The importance of $\delta$-curvatures is that, in some cases, they allow us to define $\delta$-tangents which are analogues of the tangents we defined in the finite-dimensional case.

### 4.2. **Examples.**

*Claim 4.2.* The $\delta$-curvatures of $R(r) = e^{-|r|}$ are:

$$\begin{aligned} \kappa_0 &= 1 \\ \kappa_1 &= \frac{\sqrt{2(1-e^{-\delta})}}{\delta} \\ \kappa_2 &= \frac{\sqrt{1+e^{-\delta}}}{\delta} \\ \kappa_i &= \frac{1}{\delta} \qquad \text{for } i \geq 3. \end{aligned}$$

*Proof.* The spectral function of $R$ is $d\sigma(\omega) = \frac{1}{\pi}\frac{d\omega}{1+\omega^2}$. By claim 4.1, it suffices to show that the curvatures of $\bar{\sigma}_\delta = \sigma \circ S_\delta^{-1}$ are as advertised. As explained in claim 3.11, this is equivalent to finding the Jacobi matrix associated with $\bar{\sigma}_\delta$. The latter is computed in [6, example 6.1]. □

Using a different technique, we can prove that the $\delta$-curvatures, at $t_0$, of the Brownian motion correlation, $R(t,s) = \min(t,s)$, are:

*Claim 4.3.*

$$\begin{aligned}
\kappa_0 &= \sqrt{t_0} \\
\kappa_1 &= \frac{1}{t_0}\sqrt{\frac{t_0 - \frac{1}{4}\delta}{\delta}} \\
\kappa_2 &= \frac{\sqrt{2t_0(t_0 - \frac{3}{4}\delta)}}{t_0 - \frac{1}{4}\delta}\,\frac{1}{\delta} \\
\kappa_i &= \frac{\sqrt{(t_0 - \frac{2i-5}{4}\delta)(t_0 - \frac{2i-1}{4}\delta)}}{t_0 - \frac{2i-3}{4}\delta}\,\frac{1}{\delta} \qquad \text{for } i \geq 3.
\end{aligned}$$

The proof of this claim will appear in the aforementioned paper dedicated to the study of the discrete case. It turns out that *numerically*, the $\delta$-tangents of the Brownian motion correlation about the point $t_0$ converge, as $\delta \to 0$, to an Ornstein-Uhlenbeck correlation: $\tilde{R}(r) = t_0 e^{-|r|/2t_0}$. Figure 6 provides a typical example of the quality of approximation that was observed. Note that this result agrees with our intuition, since if you normalize $R$ to have a constant variance, you obtain a time changed Ornstein-Uhlenbeck correlation:

$$\frac{R(t,s)}{\sqrt{t}\sqrt{s}} = e^{-\frac{1}{2}\left|\log(t)-\log(s)\right|},$$

and for $t, s$ near a fixed $t_0$ (cf. example 2.23),

$$e^{-\frac{1}{2}\left|\log(t)-\log(s)\right|} \sim e^{-\frac{|t-s|}{2t_0}}.$$

## 5. Somewhat technical claims and their proofs

*Claim 5.1.* Let $R$ be an s.n.d correlation and let

$$N_k(t) \stackrel{d}{=} \partial_t^k \partial_s^k R\big|_{(t,t)}.$$

Let $\{p_k\}_{k=0}^m \subset \mathbb{N}$ be a strictly decreasing sequence. Then there is a perfect equivalence between $\{N_k^{(p)}(t_0) \,:\, 0 \leq k \leq m\,,\, 0 \leq p \leq p_k\}$, and $\{\kappa_k^{(p)}(t_0) \,:\, 0 \leq k \leq m\,,\, 0 \leq p \leq p_k\}$.

*Proof.* Recall (claim 2.5) that with $\boldsymbol{x}^{(k)} = \sum_{i=0}^k c_i^k v_i$, the $c_i^k$ are polynomials in $\{\kappa_l^{(j)} \,:\, 0 \leq l \leq k\,,\, 0 \leq j \leq k-l\}$. Thus, $N_k = \sum_{i=0}^k \left(c_i^k\right)^2$, and

$$N_k^{(p)} = \sum_{i=0}^k \sum_{j=0}^p \binom{p}{j} \left(c_i^k\right)^{(j)} \left(c_i^k\right)^{(p-j)},$$

which is again a polynomial in $\{\kappa_i^{(j)} : 0 \le i \le k\,, j \le k-i+p\}$. Recalling $0 \le p \le p_k$, we find that $N_k^{(p)}$ is a polynomial in $\kappa_i^{(j)}$ with $0 \le i \le k\,, j \le p_k + k - i \le p_i$, where the last inequality is due to the monotonicity of $\{p_i\}$. This proves one half; the other implication is proved by induction on $m$.

For the base of the induction let $m = 0$. Here $N_0 = \kappa_0^2$, and

$$N_0^{(p)} = \sum_{j=1}^{p-1} \binom{p}{j} \kappa_0^{(j)} \kappa_0^{(p-j)} + 2\kappa_0 \kappa_0^{(p)},$$

so using $\kappa_0 > 0$ and increasing $p = 0, 1, \ldots, p_0$, we can determine $\kappa_0, \dot{\kappa}_0, \ddot{\kappa}_0, \ldots, \kappa_0^{(p_0)}$ consecutively.

For the inductive step, assume that $\{\kappa_k^{(p)}(t_0) : 0 \le k \le m-1\,, 0 \le p \le p_k\}$ were determined from the corresponding $\{N_k^{(p)}\}$. By claim 2.5,

$$N_m^{(p)} = \left[ \sum_{i=0}^{m-1} (c_i^m)^2 \right]^{(p)} + (\kappa_0^2 \ldots \kappa_{m-1}^2 \kappa_m^2)^{(p)}.$$

The first term is a polynomial in $\{\kappa_i^{(j)} : 0 \le i \le m-1\,, 0 \le j \le m-i+p\}$. Since $p \le p_m$, $m-i+p \le m-i+p_m \le p_i$, so this term is a polynomial in $\{\kappa_i^{(j)} : 0 \le i \le m-1\,, 0 \le j \le p_i\}$. Hence this term has been determined by the corresponding $\{N_k^{(p)}\}$. As for the second term,

$$(\kappa_0^2 \ldots \kappa_{m-1}^2 \kappa_m^2)^{(p)} = \sum_{j=1}^{p} \binom{p}{j} (\kappa_0^2 \ldots \kappa_{m-1}^2)^{(j)} (\kappa_m^2)^{(p-j)} + \kappa_0^2 \ldots \kappa_{m-1}^2 (\kappa_m^2)^{(p)}.$$

For $j \le p$, $(\kappa_0^2 \ldots \kappa_{m-1}^2)^{(j)}$ is a polynomial in $\{\kappa_k^{(l)} : k \le m-1\,, l \le p\}$, so for $p \le p_m$ it has been determined. Thus, if $m < n$, by increasing $p = 0, 1, \ldots, p_m$ we find $\kappa_m^{(p)}$ from $N_m^{(p)}$, using our inductive knowledge and the positivity of $\kappa_0, \ldots, \kappa_m$. If $m = n$, then we can determine $\kappa_n(t) = 0$ from $N_n$ (as the previous curvatures are all positive). Once we determine $\kappa_n \equiv 0$, the subsequent curvatures have to be 0 by definition. □

*Claim 5.2.* Let $\boldsymbol{x}$ be an s.n.d curve and let

$$N_k(t) \stackrel{d}{=} \langle \boldsymbol{x}_t^{(k)}, \boldsymbol{x}_t^{(k)} \rangle.$$

Then, there is a perfect equivalence between $\{N_k^{(p)}(t_0) : 0 \le k \le m\,, 0 \le p \le m-k\}$, and $\{\kappa_k^{(p)}(t_0) : 0 \le k \le m\,, 0 \le p \le m-k\}$.

*Proof.* The claim is a trivial corollary of claim 5.1 with $p_k = m - k$. □

*Proof of Claim 2.20.* By claim 5.1, it suffices to show that $\{\partial^k R|_{(t_0,t_0)} : k \le n\}$ are in perfect equivalence with $\{N_k^{(p)}(t_0) \,:\, 0 \le k \le [n/2]\,,\, 0 \le p \le n-2k\}$. In order to do so, it is convenient to change coordinates: let $\xi = (t-s)/2$ and $\eta = (t+s)/2$. As an example of the equivalence we seek to prove, consider the case $n = 2$: one can verify that,

$R(t,t) = N_0(t)$

$\partial_\xi R(t,t) = 0 \quad , \quad \partial_\eta R(t,t) = N_0'(t)$

$\partial_\xi^2 R(t,t) = N_0''(t) - 4N_1(t) \quad , \quad \partial_\xi \partial_\eta R(t,t) = 0 \quad , \quad \partial_\eta^2 R(t,t) = N_0''(t).$

Clearly, if we know $R, \partial R, \partial^2 R$ at $(t_0,t_0)$, then we know $N_0, N_0', N_0'', N_1$ at $t_0$ and vice versa. As for the general case, the identity $\partial_\xi^2 = \partial_\eta^2 - 4\partial_t\partial_s$ implies that

$$\partial_\xi^{2m} = \sum_{k=0}^{m} (-4)^k \binom{m}{k} \partial_\eta^{2m-2k} \left(\partial_t^k \partial_s^k\right).$$

If $S(t,s)$ is any smooth function of two variables then, with $\phi(t) \overset{d}{=} S(t,t)$, $\phi'(t) = \partial_\eta S\big|_{(t,t)}$, and by induction $\phi^{(k)}(t) = \partial_\eta^k S\big|_{(t,t)}$. Thus,

$$\partial_\xi^{2m} R\big|_{(t,t)} = \sum_{k=0}^{m} (-4)^k \binom{m}{k} N_k^{(2m-2k)}(t).$$

This equality holds for any smooth $R$, and by the same token

$$\partial_\eta^p \partial_\xi^{2m} R\big|_{(t,t)} = \sum_{k=0}^{m} (-4)^k \binom{m}{k} N_k^{(2m-2k+p)}(t). \tag{26}$$

If $S(t,s)$ is symmetric then $\partial_\xi S\big|_{(t,t)} = 0$, and therefore

$$\partial_\eta^p \partial_\xi^{2m+1} R\big|_{(t,t)} = \sum_{k=0}^{m} (-4)^k \binom{m}{k} \partial_\xi \left(\partial_\eta^{2m-2k+p} \partial_t^k \partial_s^k R\right)\big|_{(t,t)} = 0.$$

Let $l \in \{0, 1, \ldots, n\}$. Then, by (26), for $0 \le p \le [l/2]$,

$$\partial_\eta^{l-2p} \partial_\xi^{2p} R\big|_{(t,t)} = \sum_{k=0}^{p} (-4)^k \binom{p}{k} N_k^{(l-2k)}(t).$$

Thus, it is clear that $\{N_k^{(l-2k)}(t_0) : 0 \le k \le [l/2]\}$ determine $\{\partial_\xi^j \partial_\eta^{l-j} R|_{(t_0,t_0)} : j\}$. Conversely, given $\{\partial_\xi^j \partial_\eta^{l-j} R|_{(t_0,t_0)} : j\}$, by increasing $p$ from 0 to $[l/2]$ in the last equation, we can determine $N_p^{l-2p}(t_o)$ in this order and we are done. $\square$

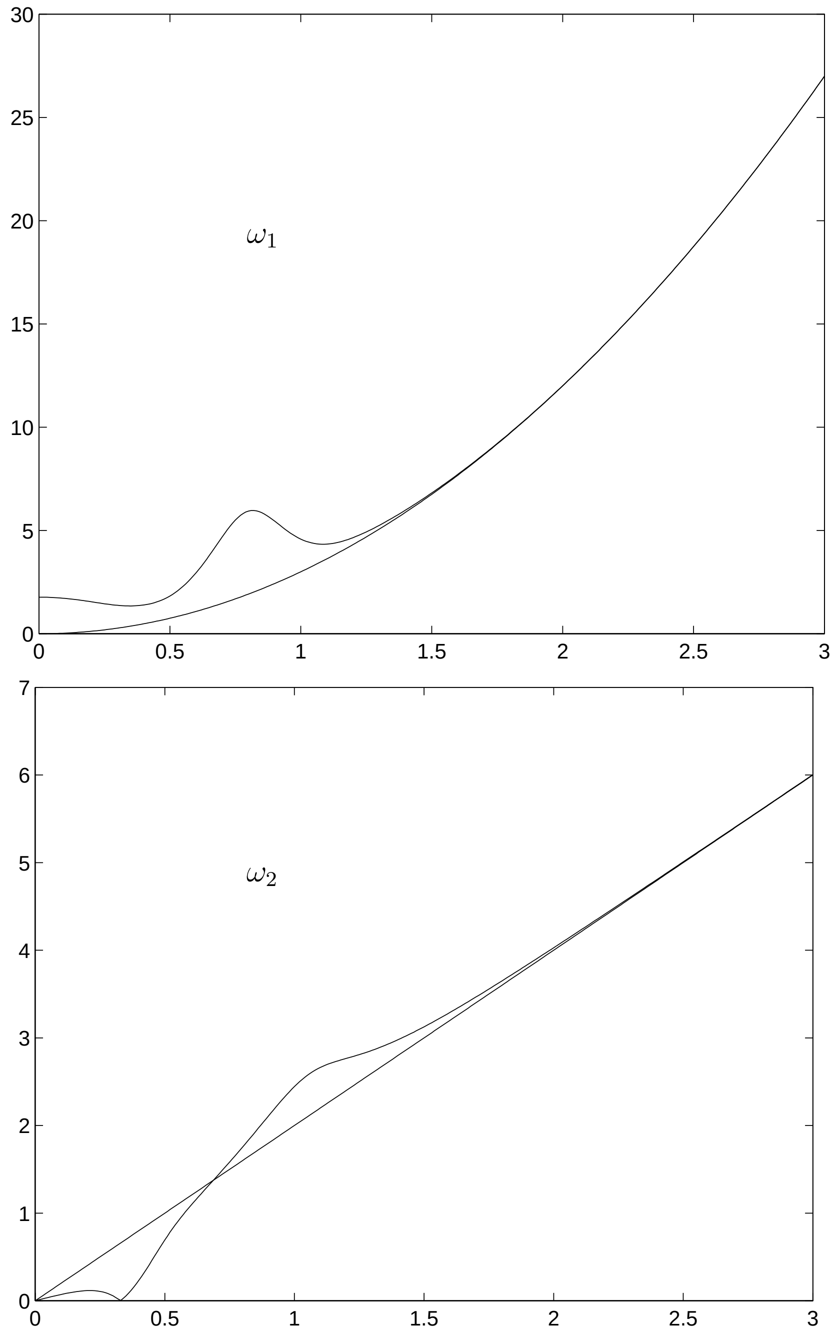


FIGURE 1. Example 2.24: frequencies of the curvature tangent.

The eigenvalues, $\omega(t_0)$, of the "symmetrized" curvature matrix, $\hat{K}(t_0)$, of $R(t,s) = \frac{1}{2}\cos(t^2 - s^2) + \frac{1}{2}\cos(t^3 - s^3)$. The first one is compared with $3t^2$ while the second with $2t$. See figure 2 for the corresponding weights.

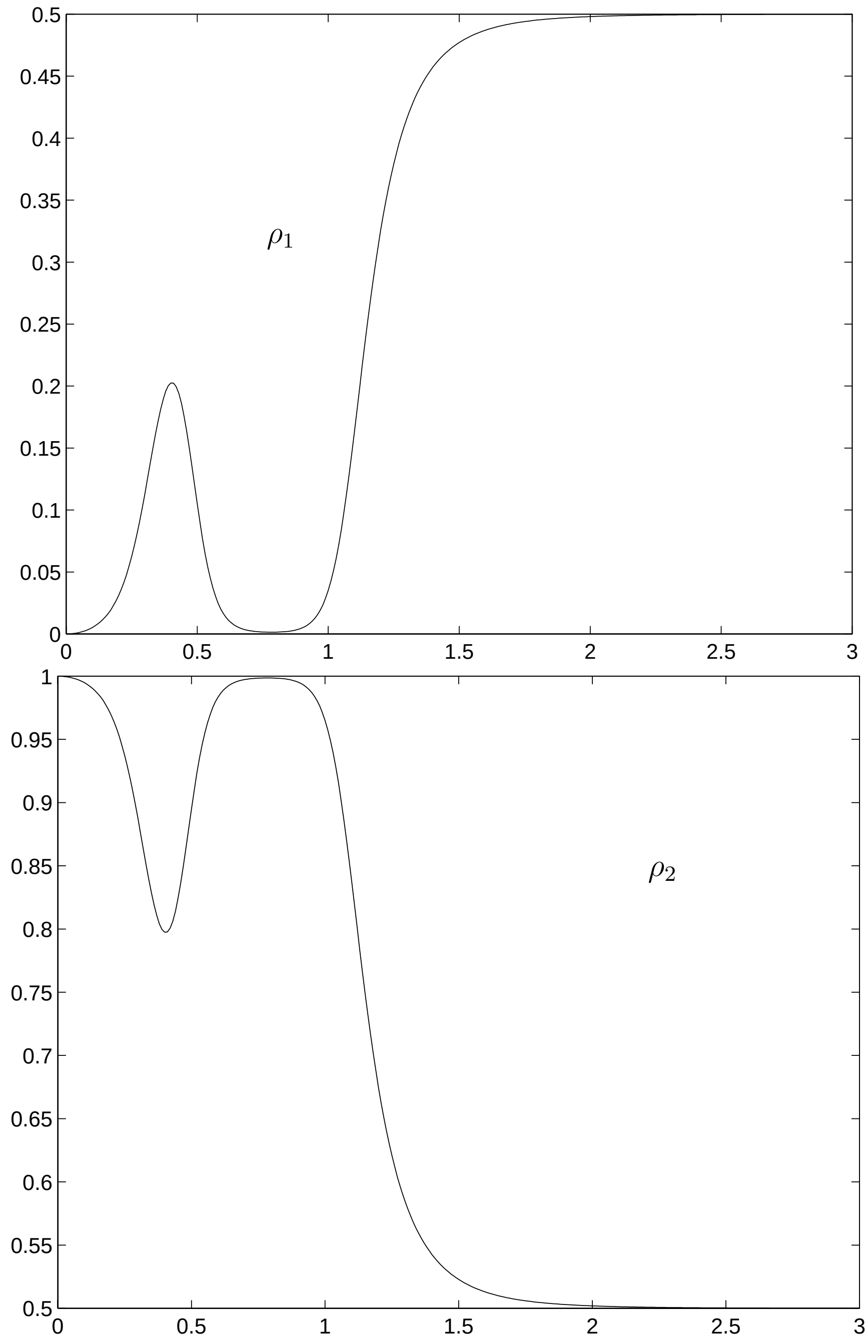


FIGURE 2. Example 2.24: radii of the curvature tangent. The weights, $\rho(t_0)$, of the spectral function associated with the curvature matrix $K(t_0)$ from the previous figure.

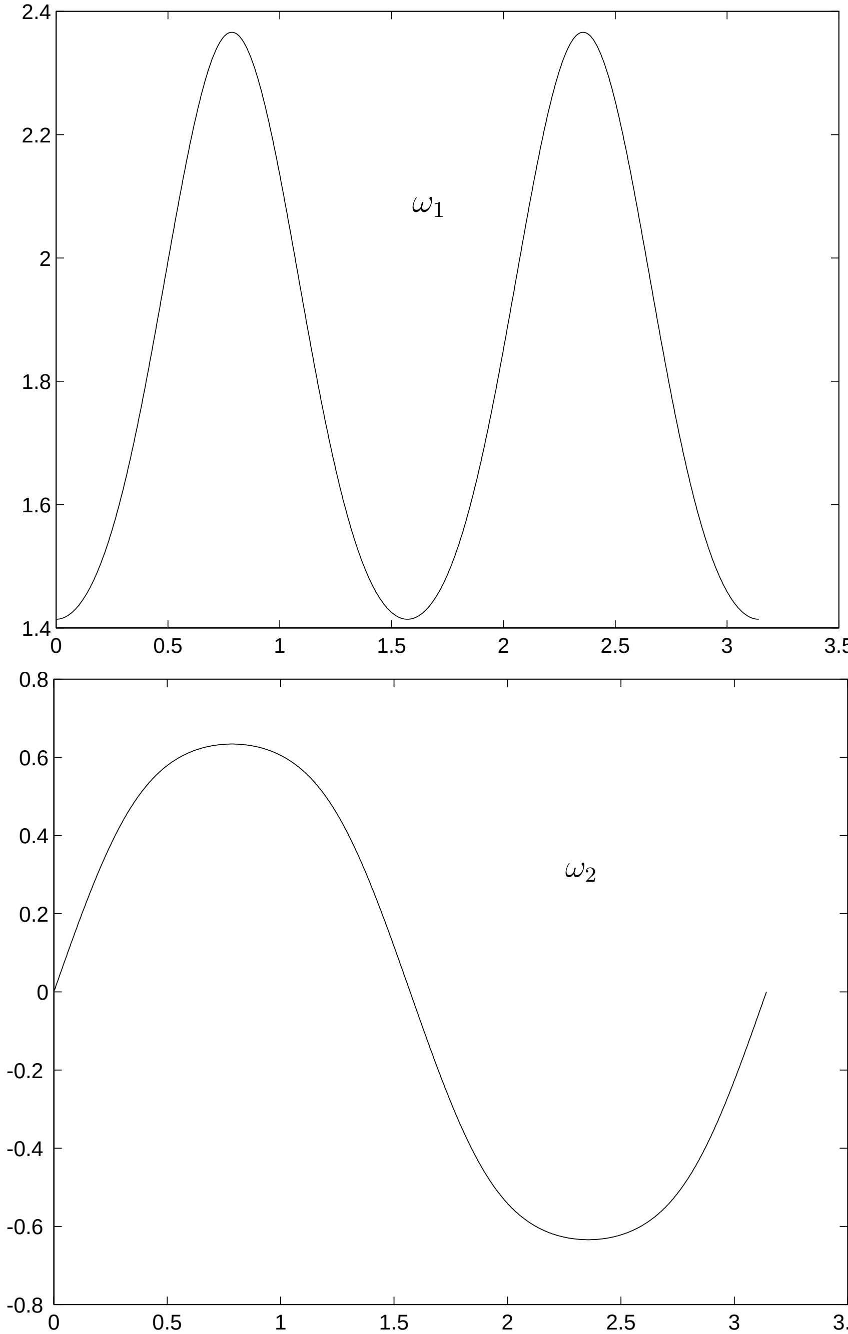


FIGURE 3. Example 2.25: frequencies of the curvature tangent.

The instantaneous "stationary frequencies", $\omega(t_0)$ (i.e. the eigenvalues of $\hat{K}(t_0)$), of $R(t,s) = \frac{1}{2}\cos\big[\cos(t) - \cos(s)\big] + \frac{1}{2}\cos\big[\sin(t) - \sin(s)\big]$. See the following figure for the corresponding radii.

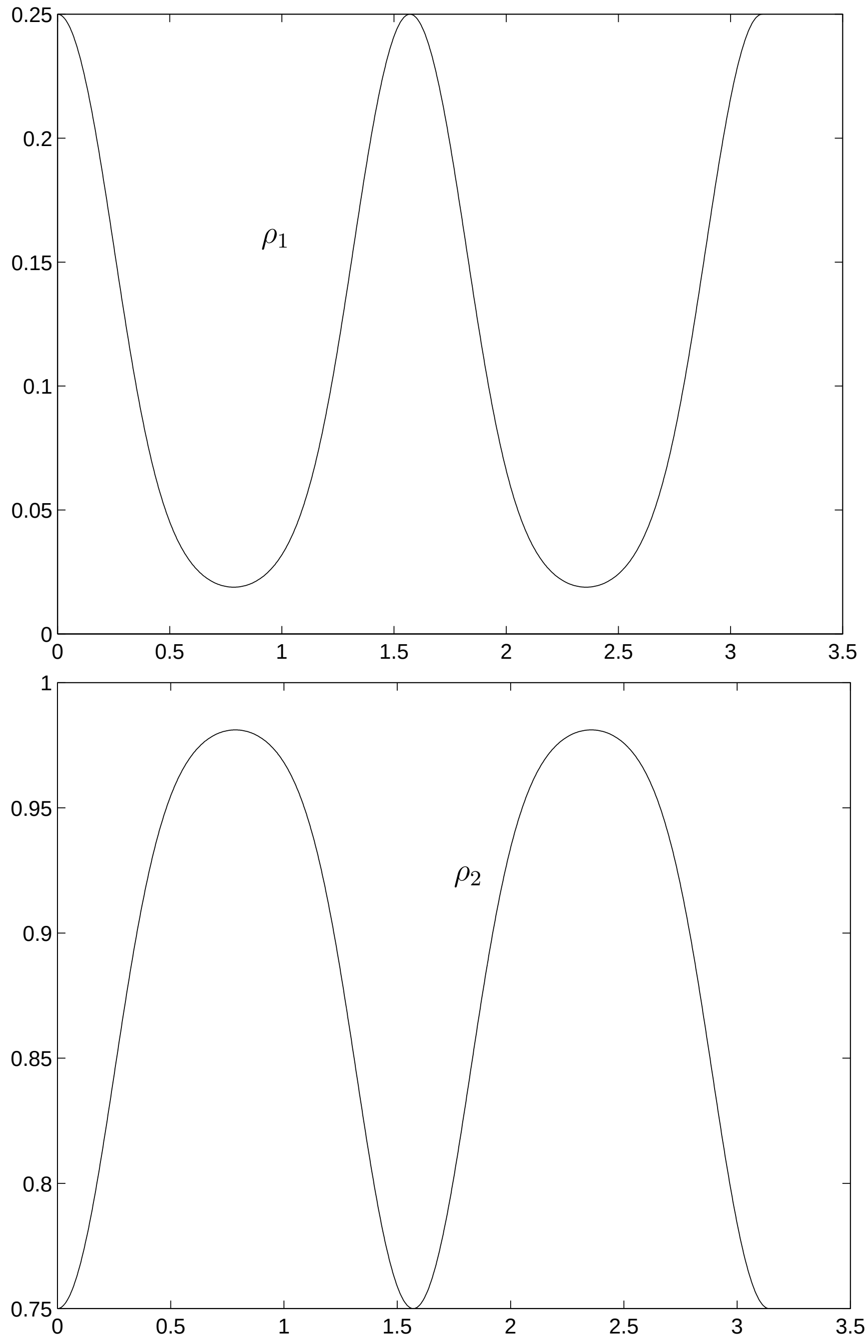


FIGURE 4. Example 2.25: radii of the curvature tangent. The weights, $\rho(t_0)$, of the tangent correlation of $R(t,s) = \frac{1}{2}\cos\left[\cos(t) - \cos(s)\right] + \frac{1}{2}\cos\left[\sin(t) - \sin(s)\right]$.

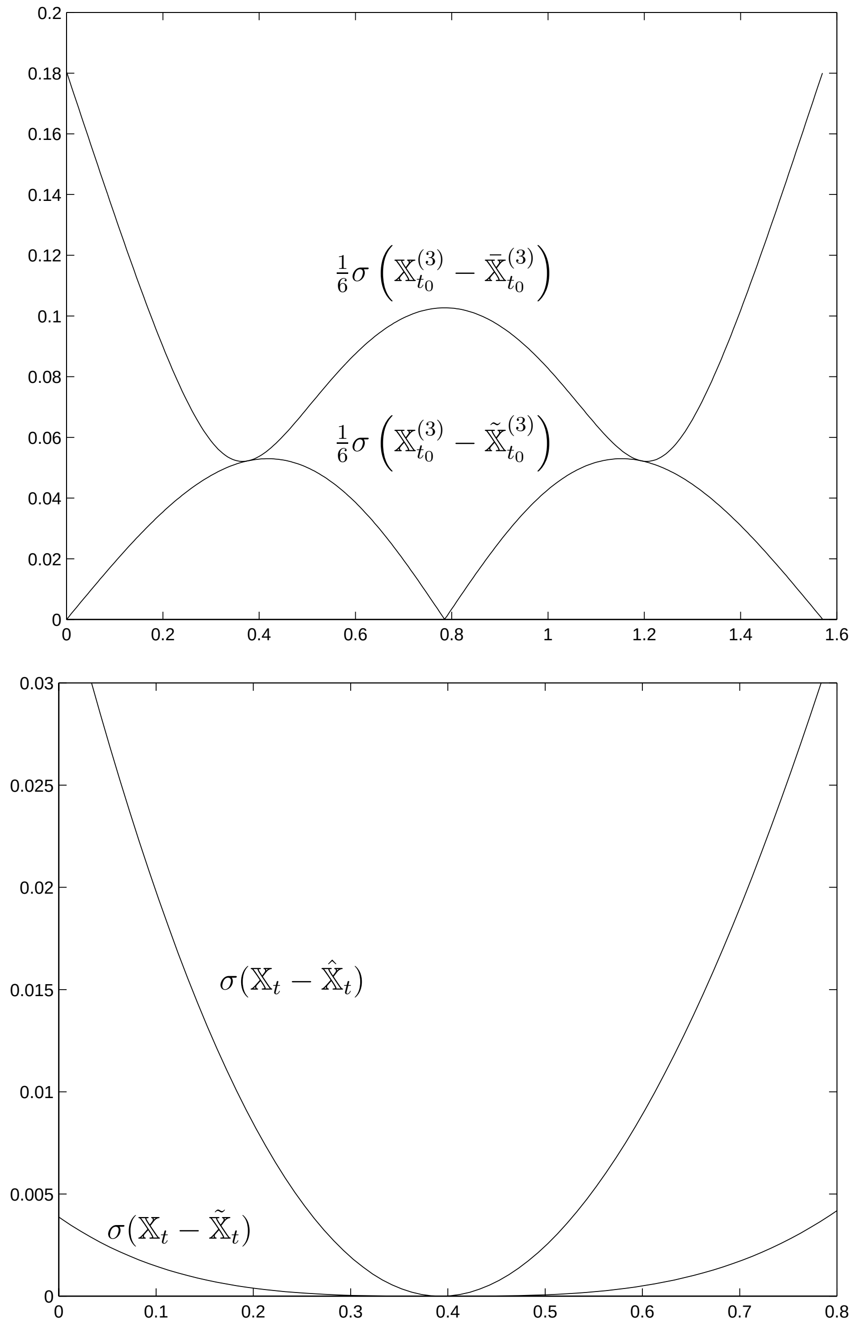


FIGURE 5. Example 2.25: comparing various approximations. The upper diagram compares the coefficients of the first non-vanishing error term in two approximations of the process $\mathbb{X}$: by $\tilde{\mathbb{X}}$, the curvature tangent, and by $\bar{\mathbb{X}}$ (has the right first 3 curvatures). The second diagram compares $\sigma(\mathbb{X}_t - \tilde{\mathbb{X}}_t)$ with $\sigma(\mathbb{X}_t - \hat{\mathbb{X}}_t)$, where $\tilde{\mathbb{X}}$ and $\hat{\mathbb{X}}$ are the tangent, respectively, the intuitive stationary approximations at $\pi/8$.

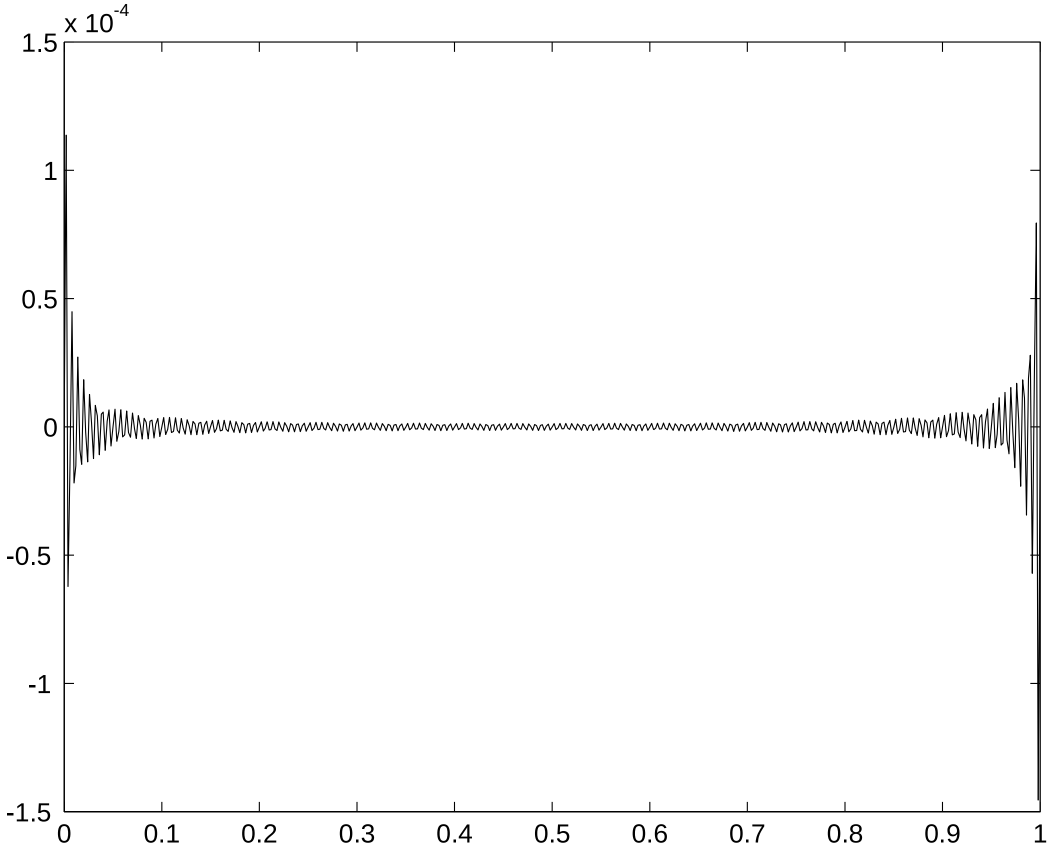


FIGURE 6. Distance between $2e^{-|r|/4}$ and the $\delta$-tangent to $t \wedge s$.

The intuitively expected tangent to the Brownian motion correlation $t \wedge s$, at $t_0 = 2$, is the correlation $\tilde{R}(r) = 2e^{-|r|/4}$. The figure depicts the relative distance between $\tilde{R}$ and the $\delta$-tangent, where $T = 1$, $n = 321$ and $\delta = \frac{T}{n-1}$. Other choices of $t_0$ yielded essentially the same picture.

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