Differential Harnack Estimates for Parabolic Equations

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Abstract Let $(M, g(t))$ be a solution to the Ricci flow on a closed Riemannian manifold. In this paper, we prove differential Harnack inequalities for positive solutions of nonlinear parabolic equations of the type

$$\frac{\partial}{\partial t} f = \Delta f - f \ln f + Rf.$$

We also comment on an earlier result of the first author on positive solutions of the conjugate heat equation under the Ricci flow.

1 Introduction

Let $(M, g(t))$, $t \in [0, T)$, be a solution to the Ricci flow on a closed manifold M. In the first part of this paper, we deal with positive solutions of nonlinear parabolic equations on M. We establish Li-Yau type differential Harnack inequalities for such positive solutions. More precisely, $g(t)$ evolves under the Ricci flow

$$\frac{\partial g(t)}{\partial t} = -2Rc, \tag{1}$$

where Rc denotes the Ricci curvature of $g(t)$. We first assume that the initial metric $g(0)$ has nonnegative curvature operator, which implies that for all time $t \in [0, T)$, $g(t)$ has nonnegative curvature operator (for example, in the case that dimension is

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4, see [7]). Consider a positive function $f(x,t)$ defined on $M \times [0,T)$, which solves the following nonlinear parabolic equation,

$$\frac{\partial f}{\partial t} = \triangle f - f \ln f + Rf, \tag{2}$$

where the symbol $\triangle$ stands for the Laplacian of the evolving metric $g(t)$ and R is the scalar curvature of $g(t)$. For simplicity, we omit $g(t)$ in the above notations. All geometry operators are with respect to the evolving metric $g(t)$.

Differential Harnack inequalities were originated by P. Li and S.-T. Yau in [12] for positive solutions of the heat equation (therefore also known as Li-Yau type Harnack estimates). The technique was then brought into the study of geometric evolution equation by R. Hamilton (for example, see [8]) and has ever since been playing an important role in the study of geometric flows. Applications include estimates on the heat kernel; curvature growth control; understanding the ancient solutions for geometric flows; proving noncollapsing result in the Ricci flow ([17]); etc. See [16] for a recent survey on this subject by L. Ni.

Using maximum principle, one can see that the solution for (2) remains positive along the flow. It exists as long as the solution for (1) exists. The study of the Ricci flow coupled with a heat-type (or backward heat-type) equation started from R. Hamilton's paper [9]. Recently, there has been some interesting study on this topic. In [17], G. Perelman proved a differential Harnack inequality for the fundamental solution of the conjugate heat equation under the Ricci flow. In [2], the first author proved a differential Harnack inequality for general positive solutions of the conjugate heat equation, which was also proved independently by S. Kuang and Q. S. Zhang in [11]. The study has also been pursued in [3, 6, 15, 20]. Various estimates are obtained recently by M. Bailesteanu, A. Pulemotov and the first author in [1], and by S. Liu in [13]. For nonlinear parabolic equations under the Ricci flow, local gradient estimates for positive solutions of equation

$$\frac{\partial}{\partial t} f = \Delta f + af \ln f + bf,$$

where a and b are constants, have been studied by Y. Yang in [19]. For general evolving metrics, similar estimate has been obtained by A. Chau, L.-F. Tam and C. Yu in [4], by S.-Y. Hsu in [10], and by J. Sun in [18]. In [14], L. Ma proved a gradient estimate for the elliptic equation

$$\Delta f + af \ln f + bf = 0.$$

In (2), if one defines

$$u(x,t) = -\ln f(x,t),$$

then the function $u = u(x,t)$ satisfies the following evolution equation

$$\frac{\partial u}{\partial t} = \Delta u - |\nabla u|^2 - R - u. \tag{3}$$

The computation from (2) to (3) is standard, which also gives the explicit relation between these two equations.

Our motivation to study (2) under the Ricci flow comes from the geometric interpretation of (3), which arises from the study of expanding Ricci solitons. Recall that given a gradient expanding Ricci soliton (M,g) satisfying

$$R_{ij} + \nabla_i \nabla_j w = -\frac{1}{4} g_{ij},$$

where w is called soliton potential function, we have

$$R(g) + \Delta_g w = -\frac{n}{4}.$$

In sight of this, by taking covariant derivative for the soliton equation and applying the second Bianchi identity, one can see that

$$R(g) + |\nabla_g w|_g^2 + \frac{w}{2} = \text{constant}.$$

Also notice that the Ricci soliton potential function w can be differed by a constant in the above equations. So by choosing this constant properly, we have

$$R(g) + |\nabla_g w|_g^2 = -\frac{w}{2} - \frac{n}{8}.$$

One consequence of the above identities is the following

$$|\nabla_g w|_g^2 = \Delta_g w - |\nabla_g w|_g^2 - R(g) - w. \tag{4}$$

Recall that the Ricci flow solution for an expanding soliton is $g(t) = c(t) \cdot \phi(t)^* g$ (c.f. [5]), where $c(t) = 1 + \frac{t}{2}$ and the family of diffeomorphism $\phi(t)$ satisfies, for any $x \in M$,

$$\frac{\partial}{\partial t}(\phi(t)(x)) = \frac{1}{c(t)} \cdot (\nabla_g w)(\phi(t)(x)).$$

Thus the corresponding Ricci soliton potential $\phi(t)^* w$ satisfies

$$\frac{\partial \phi(t)^* w}{\partial t}(x) = \frac{1}{c(t)} (\nabla_g w)(w)(\phi(t)(x)) = |\nabla \phi(t)^* w|^2(x).$$

Along the Ricci flow, (4) becomes

$$|\nabla \phi^* w|^2 = \Delta \phi^* w - |\nabla \phi^* w|^2 - R - \frac{\phi^* w}{c(t)}.$$

Hence the evolution equation for the Ricci soliton potential is

$$\frac{\partial \phi(t)^* w}{\partial t} = \Delta \phi^* w - |\nabla \phi^* w|^2 - R - \frac{\phi^* w}{c(t)}. \tag{5}$$

The second nonlinear parabolic equation that we investigate in this paper is

$$\frac{\partial u}{\partial t} = \Delta u - |\nabla u|^2 - R - \frac{u}{1+\frac{t}{2}}. \tag{6}$$

Notice that (3) and (6) are closely related and only differ by their last terms.

Our first result deals with (2) and (3).

Theorem 1. *Let $(M, g(t))$, $t \in [0, T)$, be a solution to the Ricci flow on a closed manifold, and suppose that $g(0)$ (and so $g(t)$) has weakly positive curvature operator. Let f be a positive solution to the heat equation (2), $u = -\ln f$ and*

$$H = 2\triangle u - |\nabla u|^2 - 3R - \frac{2n}{t}. \tag{7}$$

Then for all time $t \in (0, T)$

$$H \leqslant \frac{n}{4}.$$

Remark 1. The result can be generalized to the context of M being non-compact. In order for the same argument to work, we need to assume that the Ricci flow solution $g(t)$ is complete with the curvature and all the covariant derivatives being uniformly bounded and the solution u and its derivatives up to the second order are uniformly bounded *(*in the space direction*)* .

Our next result deals with (6), which is also a natural evolution equation to consider with, by the previous motivation.

Theorem 2. *Let $(M, g(t))$, $t \in [0, T)$, be a solution to the Ricci flow on a closed manifold, and suppose that $g(0)$ (and so $g(t)$) has weakly positive curvature operator. Let u be a smooth solution to (6), and define*

$$H = 2\triangle u - |\nabla u|^2 - 3R - \frac{2n}{t}. \tag{8}$$

Then for all time $t \in (0, T)$

$$H \leqslant 0.$$

Remark 2. If f is a positive function such that $f = e^{-u}$, then f satisfies the following evolution equation

$$\frac{\partial f}{\partial t} = \triangle f + Rf - \frac{f \ln f}{1+\frac{t}{2}}.$$

In [2], the first author studied the conjugate heat equation under the Ricci flow. In particular, the following theorem was proved.

Theorem 3. *[2, Theorem 3.6] Let $(M, g(t))$, $t \in [0, T]$, be a solution to the Ricci flow, and suppose that $g(t)$ has nonnegative scalar curvature. Let f be a positive solution of the conjugate heat equation*

$$\frac{\partial}{\partial t} f = -\triangle f + Rf.$$

Set $v = -\ln f - \frac{n}{2}\ln(4\pi\tau)$, $\tau = T - t$ and

$$P = 2\triangle v - |\nabla v|^2 + R - \frac{2n}{\tau}.$$

Then we have

$$\frac{\partial}{\partial \tau} P = \triangle P - 2\nabla P \cdot \nabla v - 2|v_{ij} + R_{ij} - \frac{1}{\tau} g_{ij}|^2 - \frac{2}{\tau} P - 2\frac{|\nabla v|^2}{\tau} - 2\frac{R}{\tau}. \tag{9}$$

Moreover, for all time $t \in [0, T)$,

$$P \leqslant 0.$$

In the last section, we apply a similar trick as in the proof of Theorem 1 and obtain a slightly different result, where we no longer needs to assume that $g(t)$ has nonnegative scalar curvature.

2 Proof of Theorem 1 and Application

The evolution equation of u is very similar to what is considered in [3]. So the computation for the very general setting there can be applied.

Proof (Theorem 1). In sight of the definition of H from (8) and comparing with [3, Corollary 2.2], we have

$$\frac{\partial}{\partial t}(\triangle u) = \triangle(\triangle u) - \triangle(|\nabla u|^2) - \triangle R + 2R_{ij}u_{ij} - \Delta u,$$

$$\frac{\partial}{\partial t}|\nabla u|^2 = \triangle(|\nabla u|^2) - 2|\nabla\nabla u|^2 - 2\nabla u \cdot \nabla(|\nabla u|^2) - 2\nabla u \cdot \nabla R - 2|\nabla u|^2.$$

In fact, one can directly apply the computation result there with the only modification because of the extra terms coming from time derivative $\frac{\partial}{\partial t} u$, which are put at the end of the right hand side in the above equalities. Then we have

$$\begin{aligned} \frac{\partial}{\partial t} H = & \triangle H - 2\nabla H \cdot \nabla u - 2|u_{ij} - R_{ij} - \frac{1}{t} g_{ij}|^2 - \frac{2}{t} H - \frac{2}{t}|\nabla u|^2 \\ & -2\left(\frac{\partial}{\partial t} R + \frac{R}{t} + 2\nabla R \cdot \nabla u + 2R_{ij}u_i u_j\right) - 2\Delta u + 2|\nabla u|^2, \end{aligned} \tag{10}$$

where the last two terms of the right hand side coming from the extra term $-u$ in (3). Plugging in $-2\Delta u+2|\nabla u|^2=-H+|\nabla u|^2-3R-\frac{2n}{t}$, one arrives at

$$\begin{aligned}\frac{\partial}{\partial t}H =&\triangle H-2\nabla H\cdot\nabla u-2|u_{ij}-R_{ij}-\frac{1}{t}g_{ij}|^2-\left(\frac{2}{t}+1\right)H \\ &+\left(1-\frac{2}{t}\right)|\nabla u|^2-3R-\frac{2n}{t}-\left(\frac{\partial}{\partial t}R+\frac{R}{t}+2\nabla R\cdot\nabla u+2R_{ij}u_iu_j\right).\end{aligned} \tag{11}$$

In sight of the definition of H (8), for t small enough, we have $H<0$. Since g_{ij} has weakly positive curvature operator, by the trace Harnack inequality for the Ricci flow proved by R. Hamilton in [8], we have

$$\frac{\partial}{\partial t}R+\frac{R}{t}+2\nabla R\cdot\nabla u+2R_{ij}u_iu_j\geqslant 0.$$

Also we have $R\geqslant 0$. Notice that the term $\left(1-\frac{2}{t}\right)|\nabla u|^2$ prevents us from obtaining an upper bound for H for $t>2$.

We can deal with this by the following simple manipulation. To begin with, one observes that from the definition of H,

$$|\nabla u|^2=2\left(\Delta u-R-\frac{n}{t}\right)-H-R.$$

We also have the following equality from definition,

$$tr\left(u_{ij}-R_{ij}-\frac{1}{t}g_{ij}\right)=\Delta u-R-\frac{n}{t}.$$

Now we can continue the computation for the evolution of H as follows,

$$\begin{aligned}\frac{\partial}{\partial t}H \leqslant&\triangle H-2\nabla H\cdot\nabla u-2|u_{ij}-R_{ij}-\frac{1}{t}g_{ij}|^2-\left(\frac{2}{t}+1\right)H-\frac{2}{t}|\nabla u|^2 \\ &-4R+2\left(\Delta u-R-\frac{n}{t}\right)-H-\frac{2n}{t} \\ \leqslant&\triangle H-2\nabla H\cdot\nabla u-\frac{2}{n}\left(\Delta u-R-\frac{n}{t}\right)^2-\left(\frac{2}{t}+1\right)H-\frac{2}{t}|\nabla u|^2 \\ &-4R+2\left(\Delta u-R-\frac{n}{t}\right)-H-\frac{2n}{t} \\ =&\triangle H-2\nabla H\cdot\nabla u-\left(\frac{2}{t}+2\right)H-\frac{2}{t}|\nabla u|^2-4R-\frac{2n}{t} \\ &-\frac{2}{n}\left(\Delta u-R-\frac{n}{t}-\frac{n}{2}\right)^2+\frac{n}{2} \\ \leqslant&\triangle H-2\nabla H\cdot\nabla u-\left(\frac{2}{t}+2\right)H-\frac{2}{t}|\nabla u|^2-4R-\frac{2n}{t}+\frac{n}{2}.\end{aligned}$$

The essential step is the second inequality where we make use of the elementary inequality

$$|u_{ij}-R_{ij}-\frac{1}{t}g_{ij}|^2 \geqslant \frac{1}{n}\left(\Delta u - R - \frac{n}{t}\right)^2.$$

Now we can apply maximum principle. The value of H for very small positive t is clearly very negative. So we only need to consider the maximum value point is at $t > 0$ for the desired estimate.

For $\forall T_0 < T$, assume that the maximum in $(0, T_0]$ is taken at $t_0 > 0$. At the maximum value point, using the nonnegativity of $|\nabla u|^2$ and R, one has

$$H \leqslant \frac{-4n + nt_0}{4+4t_0} = \frac{n}{4}\left(1 - \frac{5}{t_0+1}\right) \leqslant \frac{n}{4}\left(1-\frac{5}{T+1}\right).$$

So if $T \leqslant 4$, i.e., for time in $[0,4)$, $H \leqslant 0$. In general, we have

$$H \leqslant \frac{n}{4}.$$

Theorem 1 is thus proved. □

As a consequence of Theorem 1, we have

Corollary 1. *Let $(M, g(t))$, $t \in [0,T)$, be a solution to the Ricci flow on a closed manifold, and suppose that $g(0)$ (and so $g(t)$) has weakly positive curvature operator. Let f be a positive solution to the heat equation*

$$\frac{\partial}{\partial t} f = \Delta f - f \ln f + Rf.$$

Assume that (x_1,t_1) and (x_2,t_2), $0 < t_1 < t_2$, are two points in $M \times (0,T)$. Let

$$\Gamma = \inf_{\gamma} \int_{t_1}^{t_2} e^t \left(|\dot{\gamma}|^2 + R + \frac{2n}{t} + \frac{n}{4}\right) dt,$$

where γ is any space-time path joining (x_1,t_1) and (x_2,t_2). Then we have

$$e^{t_1} \ln f(x_1,t_1) \leqslant e^{t_2} \ln f(x_2,t_2) + \frac{\Gamma}{2}.$$

This inequality is in the type of classical Harnack inequalities. The proof is quite standard by integrating the differential Harnack inequality. We include it here for completeness.

Proof. Pick a space-time curve connecting (x_1,t_1) and (x_2,t_2), $\gamma(t) = (x(t),t)$ for $t \in [t_1,t_2]$. Recall that $u(x,t) = -\ln f(x,t)$. Using the evolution equation for u, we have

$$\begin{aligned}\frac{d}{dt}u\left(x(t),t\right) &= \frac{\partial u}{\partial t} + \nabla u \cdot \dot{\gamma} \\ &= \Delta u - |\nabla u|^2 - R - u + \nabla u \cdot \dot{\gamma} \\ &\leqslant \Delta u - \frac{|\nabla u|^2}{2} - R - u + \frac{|\dot{\gamma}|^2}{2}.\end{aligned} \tag{12}$$

Now by Theorem 1, we have

$$\Delta u = \frac{1}{2}\left(H + |\nabla u|^2 + 3R + \frac{2n}{t}\right) \leqslant \frac{1}{2}\left(\frac{n}{4} + |\nabla u|^2 + 3R + \frac{2n}{t}\right).$$

So we have the following estimation,

$$\frac{d}{dt}u\left(x(t),t\right) \leqslant \frac{1}{2}\left(|\dot{\gamma}|^2 + R + \frac{2n}{t} + \frac{n}{4}\right) - u.$$

For any space-time curve γ, we arrives at

$$\frac{d}{dt}(e^t \cdot u) \leqslant \frac{e^t}{2}\left(|\dot{\gamma}|^2 + R + \frac{2n}{t} + \frac{n}{4}\right).$$

Hence the desired Harnack inequality is proved by integrating t from t_1 to t_2. □

3 Proof of Theorem 2

In this section we study u satisfying the evolution equation (6) originated from gradient expanding Ricci soliton equation. We investigate the same quantity

$$H = 2\triangle u - |\nabla u|^2 - 3R - \frac{2n}{t}$$

as in the last section. The evolution equation of u, is still very similar to what is considered in [3]. We have slightly different terms coming from time derivative $\frac{\partial}{\partial t}u$ when computing the evolution equation satisfied by H. Comparing with [3, Corollary 2.2], we proceed as follows.

Proof (Theorem 2). Direct computation gives the following equation. The modification from the computation of the reference is minor as illustrated in the proof of Theorem 1.

$$\begin{aligned}\frac{\partial}{\partial t}H =& \triangle H - 2\nabla H \cdot \nabla u - 2|u_{ij} - R_{ij} - \frac{1}{t}g_{ij}|^2 - \frac{2}{t}H - \frac{2}{t}|\nabla u|^2 \\ &- 2\left(\frac{\partial}{\partial t}R + \frac{R}{t} + 2\nabla R \cdot \nabla u + 2R_{ij}u_iu_j\right) + \frac{2}{t+2}\left(-2\Delta u + 2|\nabla u|^2\right),\end{aligned} \tag{13}$$

where the last two terms of the right hand side come from the extra term $-\frac{u}{1+\frac{t}{2}}$ in (6). Plugging in $-2\Delta u+2|\nabla u|^2=-H+|\nabla u|^2-3R-\frac{2n}{t}$, one arrives at

$$\begin{aligned}\frac{\partial}{\partial t}H =& \triangle H-2\nabla H\cdot\nabla u-2|u_{ij}-R_{ij}-\frac{1}{t}g_{ij}|^2-\left(\frac{2}{t}+\frac{2}{t+2}\right)H-\frac{6}{t+2}R \\ &+\left(\frac{2}{t+2}-\frac{2}{t}\right)|\nabla u|^2-\frac{4n}{t^2+2t}-2\left(\frac{\partial}{\partial t}R+\frac{R}{t}+2\nabla R\cdot\nabla u+2R_{ij}u_iu_j\right).\end{aligned} \tag{14}$$

By the definition of H, for t small enough, we have $H<0$. Since $g(t)$ has weakly positive curvature operator, by the trace Harnack inequality for the Ricci flow ([8]), we have

$$\frac{\partial}{\partial t}R+\frac{R}{t}+2\nabla R\cdot\nabla u+2R_{ij}u_iu_j\geqslant 0.$$

Notice that now the coefficient for $|\nabla u|^2$ on the right hand side is $\frac{2}{t+2}-\frac{2}{t}<0$, and we have $R\geqslant 0$. So one can conclude directly from maximum principle that $H\leqslant 0$. □

4 A Remark on the Conjugate Heat Equation

In this section we point out a simple observation for [2, Theorem 3.6]. The assumption on scalar curvature is not needed below. We follow the original set-up in [2].

Over a closed manifold M^n, $g(t)$ for $t\in[0,T]$ is a solution to the Ricci flow (1), and $f(\cdot,t)$ is a positive solution of the conjugate heat equation

$$\frac{\partial f}{\partial t}=-\Delta f+Rf, \tag{15}$$

where Δ and R are Laplacian and scalar curvature with respect to the evolving metric $g(t)$. Notice that $\int_M f(\cdot,t)d\mu_{g(t)}$ is a constant along the flow.

Set $v=-\log f-\frac{n\log(4\pi\tau)}{2}$, where $\tau=T-t$ and define

$$P:=2\Delta v-|\nabla v|^2+R-\frac{2n}{\tau}.$$

Now we can prove the following result which is closely related to [2, Theorem 3.6].

Theorem 4. *Let $(M,g(t))$, $t\in[0,T]$, be a solution to the Ricci flow on a closed manifold. f is a positive solution to the conjugate heat equation* (15)*, and v is defines as above. Then we have*

$$\max_M\ (2\Delta v-|\nabla v|^2+R)$$

increases along the Ricci flow.

Proof. The exact computation in [2, Theorem 3.6] gives

$$\frac{\partial P}{\partial \tau} = \Delta P - 2\nabla P \cdot \nabla v - 2|\nabla^2 v + Rc - \frac{1}{\tau} g|^2 - \frac{2}{\tau} P - \frac{2}{\tau}|\nabla v|^2 - \frac{2}{\tau} R.$$

Applying the elementary inequality

$$|\nabla^2 v + Rc - \frac{1}{\tau} g|^2 \geqslant \frac{1}{n}\left(\Delta v + R - \frac{n}{\tau}\right)^2,$$

and noticing that

$$P + |\nabla v|^2 + R = 2\left(\Delta v + R - \frac{n}{\tau}\right),$$

we arrive at

$$\begin{aligned} \frac{\partial P}{\partial \tau} &\leqslant \Delta P - 2\nabla P \cdot \nabla v - \frac{1}{2n}(P + |\nabla v|^2 + R)^2 - \frac{2}{\tau}(P + |\nabla v|^2 + R) \\ &= \Delta P - 2\nabla P \cdot \nabla v - \frac{1}{2n}\left(P + |\nabla v|^2 + R + \frac{2n}{\tau}\right)^2 + \frac{2n}{\tau^2}. \end{aligned}$$

Thus if one defines

$$\widetilde{P} := P + \frac{2n}{\tau} = 2\Delta v - |\nabla v|^2 + R,$$

we have

$$\frac{\partial \widetilde{P}}{\partial \tau} \leqslant \Delta \widetilde{P} - 2\nabla \widetilde{P} \cdot \nabla v.$$

Hence $\max_M(2\Delta v - |\nabla v|^2 + R)$ decreases as τ increases, which means that it increases as t increases. This concludes the proof. □

Remark 3. Notice that we do not need to introduce τ in *Theorem 4*, but we keep the notation here so it is easy to be compared with [2, Theorem 3.6].

Remark 4. Theorem 4 and [2, Theorem 3.6] estimate quantities differ by $\frac{2n}{\tau}$. Here we do not need to assume nonnegative scalar curvature as in [2, Theorem 3.6]. Moreover, one can also prove this result for complete non-compact manifolds with proper boundness assumption.

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