Scalar Curvature Bound for Kähler-Ricci Flows over Minimal Manifolds of General Type

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Abstract

In this short paper, we prove that scalar curvature is uniformly bounded for the Kähler-Ricci flow over a minimal manifold of general type. This result can be compared with the result in [6] for the positive first Chern class case. A big part of the computation works for more general situation and we keep track of that for future application.

1 Introduction and set-up

We consider the following Kähler-Ricci flow over a closed manifold X of complex dimension $n \geqslant 2$,

$$\frac{\partial \tilde{\omega}_t}{\partial t} = -\mathrm{Ric}(\tilde{\omega}_t) - \tilde{\omega}_t, \qquad \tilde{\omega}_0 = \omega_0 \tag{1.1}$$

where ω_0 is any Kähler metric.

In this short note, we are going to prove the following theorem. Some classic computations used during the process might be of more interest. Most of them are more or less quoted directly from [7] and [6].

Theorem 1.1. *Suppose X is a non-singular minimal model of general type, then the evolving Kähler metric along the above Kähler-Ricci flow has uniformly bounded scalar curvature (for all time).*

As in [8], define $\omega_t := \omega_\infty + e^{-t}(\omega_0 - \omega_\infty)$ as the background form for the flow metric, with $\omega_\infty = -\mathrm{Ric}(\Omega)$ for a smooth volume form over X, and one has $[\omega_\infty] = -c_1(X) = K_X$ cohomologically, where K_X is the canonical class of X. Then $[\tilde{\omega}_t] = [\omega_t]$ and we can assume $\tilde{\omega}_t = \omega_t + \sqrt{-1}\partial\bar{\partial}u$. The following evolution flow of the space-time function u (usually called the metric potential of the flow metric),

$$\frac{\partial u}{\partial t} = \log\frac{(\omega_t + \sqrt{-1}\partial\bar{\partial}u)^n}{\Omega} - u, \qquad u(0, \cdot) = 0, \tag{1.2}$$

would imply (1.1), the metric flow above. They are indeed equivalent to each other by the basics on the existence and uniqueness of these flows. [1]

[1]This is illustrated in [12], for example. There is very nice discussion in [2] for complete non-compact setting.

In the case of K_X being Kähler, the flow converges exponentially fast to the (unique) Kähler-Einstein metric as completely discussed in [1], [8] and [12]. Just as in [8], the degenerate case, i.e. K_X no longer Kähler, is the main focus.

Without further clarification, all the constants appearing later are positive. The same letter C might stand for different (but fixed) positive constants at different places.

To begin with, we summarize some useful computations and estimates already known (as in [8] and [12]) for the evolution equations (1.1) and (1.2), which are valid without any further assumption on the closed manifold, X.

By direct Maximum Principle argument for (1.2), we can easily see that

$$u \leqslant C \quad \text{because} \quad \omega_t^n \leqslant C \cdot \Omega,$$

as long as the flow exists.

In the following, the gradient ∇, Laplacian Δ and norm $|\cdot|$ below are always with respect to the evolving metric along the flow, $\tilde{\omega}_t$. We start with

$$\frac{\partial}{\partial t}\left(\frac{\partial u}{\partial t}\right) = \Delta\left(\frac{\partial u}{\partial t}\right) - e^{-t}\langle\tilde{\omega}_t, \omega_0 - \omega_\infty\rangle - \frac{\partial u}{\partial t},$$

which is just the t-derivative of (1.2) and has the following two transformations,

$$\frac{\partial}{\partial t}\left(e^t\frac{\partial u}{\partial t}\right) = \Delta\left(e^t\frac{\partial u}{\partial t}\right) - \langle\tilde{\omega}_t, \omega_0 - \omega_\infty\rangle,$$

$$\frac{\partial}{\partial t}\left(\frac{\partial u}{\partial t} + u\right) = \Delta\left(\frac{\partial u}{\partial t} + u\right) - n + \langle\tilde{\omega}_t, \omega_\infty\rangle, \tag{1.3}$$

where the notation $\langle\cdot,\cdot\rangle$ means taking trace of the second term with respect to the first term which is always a metric. It is indeed equivalent to taking inner product of these two terms with respect to the first (metric) term.

A proper linear combination of these equations provides the following "finite time version" of (1.3),

$$\frac{\partial}{\partial t}\left((1-e^{t-T})\frac{\partial u}{\partial t} + u\right) = \Delta\left((1-e^{t-T})\frac{\partial u}{\partial t} + u\right) - n + \langle\tilde{\omega}_t, \omega_T\rangle. \tag{1.4}$$

If one allows $T = \infty$, this naturally gives back (1.3) above by taking $e^{-\infty} = 0$. The difference of the two transformations of the t-derivative equation would give

$$\frac{\partial}{\partial t}((1-e^t)\frac{\partial u}{\partial t} + u) = \Delta((1-e^t)\frac{\partial u}{\partial t} + u) - n + \langle\tilde{\omega}_t, \omega_0\rangle,$$

which, by Maximum Principle argument, implies the "essential decreasing" of metric potential along the flow, i.e.

$$\frac{\partial u}{\partial t} \leqslant \frac{nt + C}{e^t - 1}.$$

Let's point out that this estimate only depends on the initial value of u and its upper bound along the flow. The control is uniform away from the initial time.

Another t-derivative of the potential flow equation (1.2) gives

$$\frac{\partial}{\partial t}\left(\frac{\partial^2 u}{\partial t^2}\right) = \Delta\left(\frac{\partial^2 u}{\partial t^2}\right) + e^{-t}\langle\tilde{\omega}_t, \omega_0 - \omega_\infty\rangle - \frac{\partial^2 u}{\partial t^2} - |\frac{\partial\tilde{\omega}_t}{\partial t}|^2.$$

Take summation with the first t-derivative to arrive at

$$\frac{\partial}{\partial t}\left(\frac{\partial^2 u}{\partial t^2} + \frac{\partial u}{\partial t}\right) = \Delta\left(\frac{\partial^2 u}{\partial t^2} + \frac{\partial u}{\partial t}\right) - \left(\frac{\partial^2 u}{\partial t^2} + \frac{\partial u}{\partial t}\right) - |\frac{\partial\tilde{\omega}_t}{\partial t}|^2,$$

which provides the following estimate from Maximum Principle argument,

$$\frac{\partial^2 u}{\partial t^2} + \frac{\partial u}{\partial t} \leqslant Ce^{-t}.$$

Noticing $\tilde{\omega}_t^n = e^{\frac{\partial u}{\partial t}+u}\Omega$, this estimate implies the "essential decreasing" of volume form along the flow, i.e.

$$\frac{\partial}{\partial t}\left(\frac{\partial u}{\partial t} + u\right) \leqslant Ce^{-t}, \tag{1.5}$$

and also induces, by transforming it to be $\frac{\partial}{\partial t}\left(e^t\frac{\partial u}{\partial t}\right) \leqslant C$, that

$$\frac{\partial u}{\partial t} \leqslant (Ct + C)e^{-t}.$$

Let's point out that most of these estimates depend on the initial values of $\frac{\partial^2 u}{\partial t^2}$, $\frac{\partial u}{\partial t}$ and u, which is of course not a problem for us in this work.

Let's take a look at the metric evolution equation (1.1). Rewrite it as follows,

$$\mathrm{Ric}(\tilde{\omega}_t) = -\sqrt{-1}\partial\bar{\partial}\left(u + \frac{\partial u}{\partial t}\right) - \omega_\infty.$$

Taking trace with respect to $\tilde{\omega}_t$ for (1.1) and the one above, we have

$$R = e^{-t}\langle\tilde{\omega}_t, \omega_0 - \omega_\infty\rangle - \Delta\left(\frac{\partial u}{\partial t}\right) - n = -\Delta\left(u + \frac{\partial u}{\partial t}\right) - \langle\tilde{\omega}_t, \omega_\infty\rangle,$$

where R stands for the scalar curvature of $\tilde{\omega}_t$. Using (1.3), one also has

$$R = -n - \frac{\partial}{\partial t}\left(\frac{\partial u}{\partial t} + u\right),$$

and so the estimate got for $\frac{\partial}{\partial t}(\frac{\partial u}{\partial t} + u)$ before is equivalent to the well known fact for scalar curvature along (this rescaled version of) Ricci flow (as discussed in [7]).

2 Current interest

We have proven in [8] that the flow (1.1) (or equivalently (1.2)) exists (smoothly) as long as the cohomology class $[\tilde{\omega}_t] = [\omega_t]$ remains to be Kähler, which can be grasped by simple algebraic concern. As usual, let's define

$$T = sup\{t |\ [\omega_t] \text{ is Kähler.}\}.$$

Of course, our main interest is on the case when $[\omega_T]$ is no longer Kähler where T can be either infinite or finite. From now on, $T \leqslant \infty$ unless explicitly stated otherwise. We only consider smooth solution of Kähler-Ricci flow in $[0, T) \times X$.

At this moment, we focus on the case when the smooth limiting background form $\omega_T \geqslant 0$. It should be essentially equivalent to assume $[\omega_T]$ is the pullback of a Kähler class by some holomorphic map (or even semi-ample for algebraic geometry concern).

For $T < \infty$, we already have (1.4),

$$\frac{\partial}{\partial t}\left((1-e^{t-T})\frac{\partial u}{\partial t} + u\right) = \Delta\left((1-e^{t-T})\frac{\partial u}{\partial t} + u\right) - n + \langle \tilde{\omega}_t, \omega_T \rangle$$

with the "T" in the equation chosen to be the "T" above. Using $\omega_T \geqslant 0$, by Maximum Principle, one has

$$(1-e^{t-T})\frac{\partial u}{\partial t} + u \geqslant -C.$$

As $u \leqslant C$ and $\frac{\partial u}{\partial t} \leqslant C$, we can conclude that

$$u \geqslant -C, \qquad \frac{\partial u}{\partial t} \geqslant -\frac{C}{1-e^{t-T}} \sim \frac{C}{t-T}.$$

The situation for $T = \infty$ is somewhat different. Further assuming $[\omega_\infty] = K_X$ is also big, we still have the lower bound of u.

Remark 2.1. *This is an application of the general results obtained in [8], [4], [11] and [12] by generalizing Kolodziej's L^∞ estimate for complex Monge-Ampère equation (as summarized in [5]) to the degenerate case.*

It is observed in [12] that the lower bound of u actually implies the lower bound of $\frac{\partial u}{\partial t}$ after combining with the "essential decreasing" of volume form $\tilde{\omega}_t^n$ along the flow, i.e. (1.5). More precisely, (1.5) gives the decreasing of $\frac{\partial u}{\partial t} + u + Ce^{-t}$ along the flow. As proven in [8], over a dense open subset of X, u converges smoothly as $t \to \infty$, and so the limit of $\frac{\partial u}{\partial t}$ would have to be 0. Thus the whole term, $\frac{\partial u}{\partial t} + u + Ce^{-t}$ has limit equal to the limit of u which is uniformly bounded from below in a dense open subset. The decreasing tells us that the uniform lower bound is good for all time in a dense open subset, which is then good over X because of continuity. Hence we conclude

$$\frac{\partial u}{\partial t} \geqslant -C.$$

These are the cases under consideration in this paper. We can summarize that it is always true that

$$|(1-e^{t-T})\frac{\partial u}{\partial t} + u| \leqslant C,$$

where $e^{-\infty} = 0$ if $T = \infty$.

3 Further computation

Now we carry through some classic computations for Kähler-Ricci flow in the current situation. Similar computations can be found in [7] and [6], which are also where the author learned these stuffs, although the flow and situation being considered are not exactly the same.

3.1 Parabolic Schwarz estimate

Use the following set-up as in [7]. Suppose we have a holomorphic (non-trivial) map $F : X \to Y$ between smooth closed complex manifolds, ω is a Kähler metric over Y and $\tilde{\omega}_t$ is metric under Kähler-Ricci flow (1.1) over X.

Define $\phi := \langle \tilde{\omega}_t, F^*\omega \rangle$, then direct computation gives, over $[0, T) \times X$,

$$\left(\frac{\partial}{\partial t} - \Delta\right)\phi \leqslant \phi + C\phi^2 - H, \tag{3.1}$$

where Δ is with respect to the metric $\tilde{\omega}_t$, C is related to the bisectional curvature bound of ω over Y and $H \geqslant 0$ is described as follows. Using normal coordinates locally over X and Y, with indices i, j and α, β respectively, $\phi = |F_i^\alpha|^2$ and $H = |F_{ij}^\alpha|^2$ with all the summations. Notice that the normal coordinate over X is changing along the flow with the metric. Furthermore, one has

$$\left(\frac{\partial}{\partial t} - \Delta\right)\log\phi \leqslant C\phi + 1. \tag{3.2}$$

For our application (to the proof of Theorem 1.1), the map F is coming from the class $[\omega_T]$ with Y being some projective space $\mathbb{CP}^N$, and so ω_T is $F^*\omega$ where ω is (some multiple of) Fubini-Study metric over Y.

For simplicity, define $v := (1-e^{t-T})\frac{\partial u}{\partial t} + u$ and we have $|v| \leqslant C$ from the discussion in Section 2. We have (1.4) as follows

$$\left(\frac{\partial}{\partial t} - \Delta\right)v = -n + \langle \tilde{\omega}_t, \omega_T \rangle = -n + \phi.$$

Taking a large enough positive constant A, we have the following inequality in sight of (1.4) and (3.2),

$$\left(\frac{\partial}{\partial t} - \Delta\right)(\log\phi - Av) \leqslant -C\phi + C.$$

Since v is bounded, Maximum Principle can now be used to deduce $\phi \leqslant C$, i.e.

$$\langle \tilde{\omega}_t, \omega_T \rangle \leqslant C.$$

As in [7], in complex surface case, when the corresponding map F gives a holomorphic fibration structure of X with the base and fiber spaces being of complex dimension 1, then (at least for the regular part), one has, when restricted to each fiber,

$$\frac{\tilde{\omega}_t}{\omega_0} = \frac{\tilde{\omega}_t \wedge \omega_T}{\omega_0 \wedge \omega_T} = \frac{\tilde{\omega}_t \wedge \omega_T}{\tilde{\omega}_t^2} \cdot \frac{\tilde{\omega}_t^2}{\omega_0 \wedge \omega_T} = \frac{1}{2}\langle \tilde{\omega}_t, \omega_T \rangle \cdot \frac{\tilde{\omega}_t^2}{\omega_0 \wedge \omega_T} \leqslant C,$$

where one makes use of the volume bound in the last step.

These two estimates above give us the picture that the flow metric $\tilde{\omega}_t$ will not collapse horizontally and will be bounded fiberwise as $t \to T$.

3.2 Gradient and Laplacian estimates

In this part, we deduce gradient and Laplacian estimates for v. Recall first that

$$\left(\frac{\partial}{\partial t} - \Delta\right) v = -n + \phi, \qquad \phi = \langle \tilde{\omega}_t, \omega_T \rangle. \tag{3.3}$$

Standard computations (as in [7]) then gives:

$$\left(\frac{\partial}{\partial t} - \Delta\right)(|\nabla v|^2) = |\nabla v|^2 - |\nabla\nabla v|^2 - |\nabla\bar{\nabla} v|^2 + 2\mathrm{Re}(\nabla\phi, \nabla v), \tag{3.4}$$

$$\left(\frac{\partial}{\partial t} - \Delta\right)(\Delta v) = \Delta v + (\mathrm{Ric}(\tilde{\omega}_t), \sqrt{-1}\partial\bar{\partial} v) + \Delta\phi. \tag{3.5}$$

Recall that all the ∇, Δ and $(\cdot,\cdot)$ are with respect to $\tilde{\omega}_t$ and $\nabla\bar{\nabla} v$ is $\partial\bar{\partial} v$.

Consider the quantity $\Psi = \frac{|\nabla v|^2}{C-v}$. Since v is bounded, one can easily make sure the denominator is positive, bounded from above and also away from 0. We have the following computation,

$$\begin{aligned}
&\left(\frac{\partial}{\partial t} - \Delta\right)\Psi = \left(\frac{\partial}{\partial t} - \Delta\right)\left(\frac{|\nabla v|^2}{C-v}\right) \\
&= \frac{1}{C-v} \cdot \frac{\partial}{\partial t}(|\nabla v|^2) + \frac{|\nabla v|^2}{(C-v)^2} \cdot \frac{\partial v}{\partial t} - \left(\frac{(|\nabla v|^2)_{\bar{i}}}{C-v} + \frac{v_{\bar{i}}|\nabla v|^2}{(C-v)^2}\right)_i \\
&= \frac{|\nabla v|^2}{(C-v)^2} \cdot \left(\frac{\partial}{\partial t} - \Delta\right) v + \frac{1}{C-v} \cdot \left(\frac{\partial}{\partial t} - \Delta\right)(|\nabla v|^2) - \frac{v_i \cdot (|\nabla v|^2)_{\bar{i}}}{(C-v)^2} - v_{\bar{i}} \cdot \left(\frac{|\nabla v|^2}{(C-v)^2}\right)_i \\
&= \frac{|\nabla v|^2}{(C-v)^2} \cdot \left(\frac{\partial}{\partial t} - \Delta\right) v + \frac{1}{C-v} \cdot \left(\frac{\partial}{\partial t} - \Delta\right)(|\nabla v|^2) - \frac{2\mathrm{Re}(\nabla v, \nabla|\nabla v|^2)}{(C-v)^2} - \frac{2|\nabla v|^4}{(C-v)^3}.
\end{aligned}$$

Plug equations (3.3) and (3.4) into the equation to get

$$
\begin{aligned}
&(\frac{\partial}{\partial t}-\Delta)\Psi\\
=&\frac{(-n+\phi)|\nabla v|^2}{(C-v)^2}+\frac{|\nabla v|^2-|\nabla\nabla v|^2-|\nabla\bar{\nabla}v|^2}{C-v}+\frac{2\mathrm{Re}(\nabla\phi,\nabla v)}{C-v}\\
&-\frac{2\mathrm{Re}(\nabla v,\nabla|\nabla v|^2)}{(C-v)^2}-\frac{2|\nabla v|^4}{(C-v)^3}.
\end{aligned}
\tag{3.6}
$$

We also need the computations below to transform this expression further.

$$
\begin{aligned}
|(\nabla v,\nabla|\nabla v|^2)|&=|v_i(v_j v_{\bar{j}})_{\bar{i}}|\\
&=|v_i v_{\bar{j}} v_{j\bar{i}}+v_i v_j v_{\bar{j}\bar{i}}|\\
&\leqslant|\nabla v|^2(|\nabla\nabla v|+|\nabla\bar{\nabla}v|)\\
&\leqslant\sqrt{2}|\nabla v|^2(|\nabla\nabla v|^2+|\nabla\bar{\nabla}v|^2)^{\frac{1}{2}}.
\end{aligned}
$$

$$
\nabla\Psi=\nabla\left(\frac{|\nabla v|^2}{C-v}\right)=\frac{\nabla(|\nabla v|^2)}{C-v}+\frac{|\nabla v|^2\nabla v}{(C-v)^2}.
$$

Using the bounds for ϕ and $C-v$, we have the following computation with ϵ representing small positive constant which might be different from place to place,

$$
\begin{aligned}
&(\frac{\partial}{\partial t}-\Delta)\Psi\\
\leqslant&\, C|\nabla v|^2+\epsilon\cdot|\nabla\phi|^2-C(|\nabla\nabla v|^2+|\nabla\bar{\nabla}v|^2)\\
&-(2-\epsilon)\mathrm{Re}\left(\nabla\Psi,\frac{\nabla v}{C-v}\right)-\epsilon\cdot\frac{\mathrm{Re}(\nabla v,\nabla|\nabla v|^2)}{(C-v)^2}-\epsilon\cdot\frac{|\nabla v|^4}{(C-v)^3}\\
\leqslant&\, C|\nabla v|^2+\epsilon\cdot|\nabla\phi|^2-C(|\nabla\nabla v|^2+|\nabla\bar{\nabla}v|^2)\\
&-(2-\epsilon)\mathrm{Re}\left(\nabla\Psi,\frac{\nabla v}{C-v}\right)+\epsilon\cdot(|\nabla\nabla v|^2+|\nabla\bar{\nabla}v|^2)-\epsilon\cdot|\nabla v|^4\\
\leqslant&\, C|\nabla v|^2+\epsilon\cdot|\nabla\phi|^2-(2-\epsilon)\mathrm{Re}\left(\nabla\Psi,\frac{\nabla v}{C-v}\right)-\epsilon\cdot|\nabla v|^4.
\end{aligned}
$$

We need a few more calculations to set up Maximum Principle argument. Recall that $\phi=\langle\tilde{\omega}_t,\omega_T\rangle$ and (3.1),

$$
\left(\frac{\partial}{\partial t}-\Delta\right)\phi\leqslant\phi+C\phi^2-H.
$$

With the description of H before and the estimate for ϕ, i.e. $\phi\leqslant C$, from Subsection 3.1, we can conclude by classic computation (as in [7]) that

$$
H\geqslant C|\nabla\phi|^2.
$$

Now one arrives at $\left(\frac{\partial}{\partial t} - \Delta\right)\phi \leqslant C - C|\nabla\phi|^2$, which is also, for small enough positive constant ϵ,

$$\left(\frac{\partial}{\partial t} - \Delta\right)\phi + \epsilon|\nabla\phi|^2 \leqslant C.$$

Of course, one has

$$|\left(\nabla\phi, \frac{\nabla v}{C-v}\right)| \leqslant \epsilon \cdot |\nabla\phi|^2 + C \cdot |\nabla v|^2. \tag{3.7}$$

Now consider the function $\Psi + \phi$. Choosing $\epsilon > 0$ small enough [2], we have

$$\left(\frac{\partial}{\partial t} - \Delta\right)(\Psi + \phi) \leqslant C + C|\nabla v|^2 - \epsilon \cdot |\nabla v|^4 - (2-\epsilon)\mathrm{Re}\left(\nabla(\Psi+\phi), \frac{\nabla v}{C-v}\right).$$

At the maximum value point of $\Psi + \phi$, we know $|\nabla v|^2$ can not be too large. It's then easy to conclude the upper bound for $\Psi + \phi$, and so for Ψ. Hence we have achieved the gradient estimate,

$$|\nabla v| \leqslant C.$$

Now we want to do similar things for the Laplacian, Δv. Define $\Phi := \frac{C-\Delta v}{C-v}$. Similar computation as before gives the following

$$\begin{aligned}
&\left(\frac{\partial}{\partial t} - \Delta\right)\Phi = \left(\frac{\partial}{\partial t} - \Delta\right)\left(\frac{C-\Delta v}{C-v}\right) \\
&= -\frac{1}{C-v} \cdot \left(\frac{\partial}{\partial t} - \Delta\right)\Delta v + \frac{C-\Delta v}{(C-v)^2} \cdot \left(\frac{\partial}{\partial t} - \Delta\right) v \\
&\quad + \frac{2\mathrm{Re}(\nabla v, \nabla\Delta v)}{(C-v)^2} - \frac{2|\nabla v|^2(C-\Delta v)}{(C-v)^3} \\
&= -\frac{1}{C-v} \cdot \left(\Delta v + (\mathrm{Ric}(\tilde\omega_t), \sqrt{-1}\partial\bar\partial v) + \Delta\phi\right) + \frac{C-\Delta v}{C-v} \cdot (-n + \phi) \\
&\quad + \frac{2\mathrm{Re}(\nabla v, \nabla\Delta v)}{(C-v)^2} - \frac{2|\nabla v|^2(C-\Delta v)}{(C-v)^3}.
\end{aligned}$$

We also have $\nabla(\frac{C-\Delta v}{C-v}) = \frac{(C-\Delta v)\nabla v}{(C-v)^2} - \frac{\nabla\Delta v}{C-v}$. Recall that it is already proven $(0 \leqslant)\phi \leqslant C$. The following inequality follows from standard computation (as in [7]) and has actually been used before,

$$\Delta\phi \geqslant (\mathrm{Ric}\,(\tilde\omega_t), \omega_T) + H - C\phi^2,$$

where $H \geqslant C|\nabla\phi|^2 \geqslant 0$ from the bound of ϕ as used already before. Now we have

$$(\mathrm{Ric}, \sqrt{-1}\partial\bar\partial v) + \Delta\phi \geqslant (\mathrm{Ric}, \sqrt{-1}\partial\bar\partial v + \omega_T) - C$$

[2] Of course, this affects the choices of C's, but they will all be fixed eventually.

with $\mathrm{Ric} = \mathrm{Ric}(\tilde{\omega}_t)$.

At this moment, let's finally restrict ourselves to the case of $T = \infty$ as for Theorem 1.1. Then we have

$$v = \frac{\partial u}{\partial t} + u, \qquad \mathrm{Ric} = -\sqrt{-1}\partial\bar{\partial}v - \omega_\infty.$$

Now the previous estimation can be continued as follows,

$$\begin{aligned}(\mathrm{Ric}, \sqrt{-1}\partial\bar{\partial}v) + \Delta\phi &\geqslant (\mathrm{Ric}, \sqrt{-1}\partial\bar{\partial}v + \omega_\infty) - C \\ &= -|\sqrt{-1}\partial\bar{\partial}v + \omega_\infty|^2 - C \\ &\geqslant -(1+\epsilon)|\sqrt{-1}\partial\bar{\partial}v|^2 - C\cdot|\omega_\infty|^2 - C,\end{aligned}$$

where ϵ is a small positive number. As $\phi = \langle\tilde{\omega}_t, \omega_\infty\rangle \leqslant C$ and $\omega_\infty \geqslant 0$, which give $|\omega_\infty|^2 \leqslant C$, one finally arrives at

$$(\mathrm{Ric}, \sqrt{-1}\partial\bar{\partial}v) + \Delta\phi \geqslant -(1+\epsilon)|\nabla\bar{\nabla}v|^2 - C.$$

If $T = \infty$, we also have $\Delta v = -R - \phi \leqslant C$, and so the numerator and denominator of Φ can both be positive and bounded away from 0. We only want to make sure that the numerator can not be too large.

Let's continue the computation for Φ,

$$\left(\frac{\partial}{\partial t} - \Delta\right)\Phi \leqslant C + C\cdot(C - \Delta v) + \frac{(1+\epsilon)|\nabla\bar{\nabla}v|^2}{C - v} - 2\mathrm{Re}\left(\nabla\Phi, \frac{\nabla v}{C - v}\right).$$

Now we need the computation for Ψ before, but from a slightly different point of view. There is no need to involve ϵ now. Also, it is already known that $|\nabla v| \leqslant C$. Basically from (3.6) and (3.7), we get the following inequality,

$$\left(\frac{\partial}{\partial t} - \Delta\right)(\Psi + \phi) \leqslant C - \frac{|\nabla\bar{\nabla}v|^2}{C - v} - 2\mathrm{Re}\left(\nabla(\Psi + \phi), \frac{\nabla v}{C - v}\right).$$

Combining the two inequalities above and choosing ϵ small enough, we get

$$\left(\frac{\partial}{\partial t} - \Delta\right)(\Phi + 2\Psi + 2\phi) \leqslant C + C\cdot(C - \Delta v) - C\cdot|\nabla\bar{\nabla}v|^2 - 2\mathrm{Re}\left(\nabla(\Phi + 2\Psi + 2\phi), \frac{\nabla v}{C - v}\right).$$

Using the inequality above, one only needs to observe

$$|\nabla\bar{\nabla}v|^2 \geqslant C(\Delta v)^2 \geqslant C(C - \Delta v)^2 - C$$

to carry through the Maximum Principle argument for $\Phi + 2\Psi + 2\phi$. Hence we can conclude that $\Phi \leqslant C$, and so

$$-\Delta v \leqslant C,$$

which is equivalent to $R \leqslant C$ as ϕ is bounded for our concern.

The uniform lower bound for scalar curvature is classic as in [7], and we have finished the proof of Theorem 1.1.

4 Remarks

Of course, Theorem 1.1 would imply the uniform bound of scalar curvature for the limiting metric (in the regular open dense part), which is a rather trivial result as the limiting metric is Kähler-Einstein in this case. The (local) convergence is treated in [8].

In surface case, the limiting metric is an orbifold Kähler-Einstein metric, which is more than just a metric with bounded scalar curvature over the regular part. For general dimension, hopefully this scalar curvature bound along Kähler-Ricci flow would be helpful to improve our knowledge on the limiting (singular Kähler-Einstein) metric itself.

Recently, the result here has been applied by Yuguang Zhang to provide an alternative proof of the classic Miyaoka-Yao Inequality using flow construction (see [10]).

In sight of the argument for bounded diameter in [6], one might want to get similar result for our case. There is a quite involving issue that our flow corresponds to the infinite time case for the Ricci flow (before rescaling) and so Perelman's W-functional won't be as effective for us.

One might think of applying these computations for more general flows as introduced by H. Tsuji in [9] and discussed a little more in [8] and [12]. More importantly, try to use them for this flow discussed above for finite time singularity case as the last part of the argument only works now for $T = \infty$ so far. This is very interesting and also challenging. In fact, for the case of finite time singularity, the situation is very different. We can already show the blow-up of scalar curvature when approaching the finite singular time. The argument makes use of our flow computation and the characterization of Kähler cone by Demailly and Paun (in [3]). This, together with more discussions for finite time singularity, would be included in a paper of continuation [13].

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