This paper comprises 6 questions worth 10 marks each, for a total of 60 marks. Each question is divided into several parts; the marks assigned to each part are indicated at the end of the question.

Questions 1, 2, 4, 5 are the same for MATH2069 and MATH2969. For questions 3, 6 this paper contains both the normal-level MATH2069 question and the (completely different) advanced-level MATH2969 question. You may ONLY answer the questions for the unit you are enrolled in.

If you can’t solve one part of a question, you can still assume the result in doing later parts.

No notes or books are allowed. Approved calculators are permitted.

SOLUTIONS
1. In this question you are required to evaluate your answers: it is not sufficient to merely give a formula for the answer. You must justify your answers.

Five boys Alan, Ben, Chris, Don and Ed are to be given toy cars. All cars are of the same make and only differ by colour. Calculate the number of ways to distribute

(a) four cars of different colours among the boys;
   **Solution:** This is the number of ordered selections of 4 things from 5 possibilities with repetitions allowed. Hence the answer is $5^4 = 625$.

(b) four red cars among the boys;
   **Solution:** This is the number of unordered selections of 4 things from 5 possibilities with repetitions allowed. Hence the answer is $\binom{4+5-1}{4} = 70$.

(c) eleven cars of different colours so that Alan and Ben get three cars each, Chris and Don get two cars each and Ed gets one car;
   **Solution:** The number of ways equals the multinomial coefficient $\binom{11}{3,3,2,2,1} = 277200$.

(d) five cars of different colours so that only Ben and Ed miss out;
   **Solution:** This is the number of surjective functions from a set of cardinality 5 to a set of cardinality 3. Hence the answer is $3!S(5,3) = 6 \cdot 25 = 150$.

(e) twelve blue cars so that everyone gets a different number of cars.
   **Solution:** The minimal number which can be written as the sum of 5 distinct natural numbers is 10 with $10 = 0 + 1 + 2 + 3 + 4$. This implies that there are two ways to write 12 as the sum of 5 distinct natural numbers, namely, $12 = 0 + 1 + 2 + 3 + 6$ and $12 = 0 + 1 + 2 + 4 + 5$. Taking into account $5! = 120$ permutations of each set $\{0, 1, 2, 3, 6\}$ and $\{0, 1, 2, 4, 5\}$, we find that the answer is $120 + 120 = 240$.

   $(2 + 2 + 2 + 2 + 2 = 10$ marks $)$

2. (a) Consider the recurrence relation $b_n = 2b_{n-1} + 7b_{n-2} + 4b_{n-3}$, $n \geq 3$. Given that $x = -1$ is a root of its characteristic polynomial, find all the other roots and write down the general solution of the recurrence relation.
   **Solution:** The characteristic polynomials is $x^3 - 2x^2 - 7x - 4$. Since $x = -1$ is its root, it is divisible by $x+1$. Hence $x^3 - 2x^2 - 7x - 4 = (x+1)(ax^2 + bx + c)$ for some constants $a, b$ and $c$. Comparing the coefficients of the same powers of $x$ we find that $a = 1$, $c = -4$ and $a + b = -2$. Thus, $x^3 - 2x^2 - 7x - 4 = (x+1)(x^2 - 3x - 4)$ (this can also be obtained by long division of polynomials). The roots of the equation $x^2 - 3x - 4 = 0$ are $x = -1$ and $x = 4$ so that $x = -1$ is a root of the characteristic equation of multiplicity 2 and $x = 4$ is a root of multiplicity 1. Hence the general solution of the recurrence relation is $b_n = (An + B)(-1)^n + C 4^n$, where $A, B$ and $C$ are arbitrary constants.
(b) Find the general solution of the nonhomogeneous recurrence relation
\[ a_n = 2a_{n-1} + 7a_{n-2} + 4a_{n-3} + 3^n, \quad n \geq 3. \]

**Solution:** To find a particular solution, take \( p_n = D \cdot 3^n \). Substituting in the relation we get
\[ D \cdot 3^n = 2D \cdot 3^{n-1} + 7D \cdot 3^{n-2} + 4D \cdot 3^{n-3} + 3^n. \]

Divide by \( 3^{n-3} \) and rearrange to get \( (27 - 18 - 21 - 4)D = 27 \) so that \( D = -27/16 \). Hence the general solution is
\[ a_n = (A_n + B)(-1)^n + C \cdot 4^n - \frac{27}{16} \cdot 3^n. \]

(c) Explain why it is not possible to find a particular solution of the nonhomogeneous recurrence relation
\[ a_n = 2a_{n-1} + 7a_{n-2} + 4a_{n-3} + (-1)^n, \quad n \geq 3, \]
in the form \( q_n = C(-1)^n \), where \( C \) is a constant.

**Solution:** Since \( x = -1 \) is a root of the characteristic equation, all sequences of the form \( q_n = C(-1)^n \) are solutions of the homogeneous relation. Therefore, it is not possible to find a value of \( C \) to satisfy the nonhomogeneous recurrence relation.

(d) Suppose that \( p_n \) is a particular solution of the recurrence relation in part (c) (you do not need to find \( p_n \) explicitly). Write down the general solution of the recurrence relation
\[ a_n = 2a_{n-1} + 7a_{n-2} + 4a_{n-3} + 3^{n+2} - (-1)^n, \quad n \geq 3. \]

**Solution:** By parts (a) and (b), the general solution is
\[ a_n = (A_n + B)(-1)^n + C \cdot 4^n - \frac{243}{16} \cdot 3^n - p_n. \]

\[ (3 + 3 + 2 + 2 = 10 \text{ marks}) \]

3. This question is for MATH2069 students only.

(a) Write down a closed form for the generating function of the sequence \( a_n \) defined by \( a_n = n \cdot (-2)^n + 5 \cdot 3^n. \)

**Solution:** We have
\[ A(z) = \sum_{n=0}^{\infty} a_n z^n = \sum_{n=0}^{\infty} (n \cdot (-2)^n + 5 \cdot 3^n) z^n = \sum_{n=0}^{\infty} n \cdot (-2z)^n + 5 \sum_{n=0}^{\infty} (3z)^n = -\frac{2z}{(1 + 2z)^2} + \frac{5}{1 - 3z}. \]

(b) Consider the sequence \( b_n \) given by \( b_n = (n + 1)a_{n+1} \). Using your answer to part (a) or otherwise, write a closed form for the generating function of the sequence \( b_n \).
Solution: The generating function
\[ B(z) = \sum_{n=0}^{\infty} b_n z^n = \sum_{n=0}^{\infty} (n+1) a_{n+1} z^n \]
coincides with the derivative of the generating function \( A(z) \) found in part (a). Hence by the quotient rule,
\[
B(z) = -\frac{(2z)'(1 + 2z)^2 - ((1 + 2z)^2)'2z}{(1 + 2z)^4} + \frac{(5)'(1 - 3z) - (1 - 3z)'5}{(1 - 3z)^2}
\]
\[
= -\frac{2(1 + 2z) - 8z}{(1 + 2z)^3} + \frac{15}{(1 - 3z)^2} = \frac{4z - 2}{(1 + 2z)^3} + \frac{15}{(1 - 3z)^2}.
\]

(c) The generating function \( C(z) \) for a sequence \( c_n \) is given by
\[
C(z) = \frac{1 + 2z}{1 + 5z + 4z^2}.
\]

Give an explicit formula for \( c_n \).
Solution: Use partial fraction decomposition
\[
\frac{1 + 2z}{1 + 5z + 4z^2} = \frac{1 + 2z}{(1 + z)(1 + 4z)} = \frac{A}{1 + z} + \frac{B}{1 + 4z},
\]
so that the constants \( A \) and \( B \) are found from the identity
\[
1 + 2z = A(1 + 4z) + B(1 + z).
\]
We have \( A + B = 1 \) and \( 4A + B = 2 \). Hence \( A = 1/3 \) and \( B = 2/3 \). Therefore
\[
C(z) = \frac{1}{3(1 + z)} + \frac{2}{3(1 + 4z)} = \frac{1}{3} \sum_{n=0}^{\infty} (-z)^n + \frac{2}{3} \sum_{n=0}^{\infty} (-4z)^n
\]
and so \( c_n = \frac{1}{3} (-1)^n + \frac{2}{3} (-4)^n \).

(d) Let \( c_n \) be as in part (c), and define
\[
d_n = (-2)^n c_0 + (-2)^{n-1} c_1 + \cdots + (-2) c_{n-1} + c_n
\]
for all integers \( n \geq 0 \).
Using generating functions or otherwise, find an explicit formula for \( d_n \).
Solution: Since
\[
\sum_{n=0}^{\infty} (-2)^n z^n = \frac{1}{1 + 2z},
\]
the generating function \( D(z) \) of the sequence \( d_n \) is the product of \( C(z) \) and \( \frac{1}{1 + 2z} \). Therefore
\[
D(z) = \frac{1}{1 + 5z + 4z^2} = \frac{1}{(1 + z)(1 + 4z)} = -\frac{1}{3(1 + z)} + \frac{4}{3(1 + 4z)}
\]
\[
= -\frac{1}{3} \sum_{n=0}^{\infty} (-z)^n + \frac{4}{3} \sum_{n=0}^{\infty} (-4z)^n
\]
and $d_n = -\frac{1}{3} (-1)^n + \frac{4}{3} (-4)^n$.

$(2 + 2 + 3 + 3 = 10$ marks$)$

3. This question is for MATH2969 students only.

(a) Prove the formal power series identity

$$\exp(az) \exp(bz) = \exp((a+b)z),$$

where $a$ and $b$ are constants.

**Solution:** The coefficient of $z^n$ on the left hand side equals

$$\sum_{k+l=n} \frac{a^k b^l}{k! l!} = \frac{1}{n!} \sum_{k=0}^n \binom{n}{k} a^k b^{n-k} = \frac{1}{n!} (a+b)^n$$

by the binomial theorem. This coincides with the coefficient of $z^n$ on the right hand side.

(b) Let $F(z)$ and $H(z)$ be formal power series. Consider the differential equation

$$A'(z) = F(z)A(z) + H(z)$$

for a formal power series $A(z)$. Suppose that $G(z)$ is the unique formal power series with zero constant term such that $G'(z) = F(z)$. Using the substitution $A(z) = B(z) \exp \left( G(z) \right)$, write down an explicit expression for the derivative $B'(z)$ in terms of $G(z)$ and $H(z)$.

**Solution:** Calculate $A'(z)$ by applying the product and chain rules

$$A'(z) = B'(z) \exp \left( G(z) \right) + B(z) \exp \left( G(z) \right) G'(z)$$

$$= B'(z) \exp \left( G(z) \right) + A(z) F(z).$$

Substituting into the differential equation we get

$$B'(z) \exp \left( G(z) \right) + A(z) F(z) = F(z)A(z) + H(z)$$

and so

$$B'(z) = H(z) \left[ \exp \left( G(z) \right) \right]^{-1} = H(z) \exp \left( -G(z) \right),$$

where the last equality holds by part (a) with $a + b = 0$.

(c) Using parts (a) and (b) or otherwise, find the solution of the differential equation

$$A'(z) = 5 A(z) + \exp(-3z)$$

satisfying the condition $a_0 = 0$.

**Solution:** The equation is of the form given in part (b) with $F(z) = 5$ and $H(z) = \exp(-3z)$. Hence $G(z) = 5z$ and we get the equation for $B(z)$,

$$B'(z) = \exp(-3z) \exp(-5z) = \exp(-8z)$$

**SOLUTIONS**
by parts (a) and (b). Solving this for $B(z)$ we get

$$B(z) = c - \frac{1}{8} \exp(-8z)$$

where $c$ is a constant. Then

$$A(z) = \left(c - \frac{1}{8} \exp(-8z)\right) \exp(5z) = c \exp(5z) - \frac{1}{8} \exp(-3z).$$

Since $a_0 = 0$ the value of $c$ is $1/8$ and so

$$A(z) = \frac{1}{8} \left(\exp(5z) - \exp(-3z)\right).$$

(d) Consider the sequence

$$d_n = 4^n L_0 + 4^{n-1} L_1 + \cdots + 4 L_{n-1} + L_n, \quad n \geq 0,$$

where $L_n$ are the Lucas numbers with $L_0 = 2$ and $L_1 = 1$. Write down the generating function $D(z)$ of the sequence $d_n$ in a closed form.

**Solution:** The Lucas numbers satisfy the recurrence relation $L_n = L_{n-1} + L_{n-2}$ for $n \geq 2$. Hence, for the generating function we get

$$L(z) = \sum_{n=0}^{\infty} L_n z^n = 2 + z + \sum_{n=2}^{\infty} (L_{n-1} + L_{n-2}) z^n = 2 + z + z(L(z) - 2) + z^2 L(z)$$

and so

$$L(z) = \frac{2 - z}{1 - z - z^2}.$$

Since

$$\sum_{n=0}^{\infty} 4^n z^n = \frac{1}{1 - 4z},$$

the generating function $D(z)$ of the sequence $d_n$ is the product of $L(z)$ and $\frac{1}{1 - 4z}$. Therefore

$$D(z) = \frac{2 - z}{(1 - 4z)(1 - z - z^2)}.$$

(2 + 2 + 3 + 3 = 10 marks)

4. (a) State the theorem from lectures which provides a necessary and sufficient condition for a connected graph to be Eulerian.

**Solution:** A connected graph $G$ is Eulerian if and only if the degree $\deg(v)$ is even for every vertex $v$ of $G$. 

SOLUTIONS turn to page 7
(b) Apply the theorem in part (a) to conclude that the following graph is not Eulerian.

![Graph](image)

**Solution:** The degree of vertex $a$ is 3 which is an odd number. Therefore, the graph is not Eulerian.

(c) What is the minimum number of edges which can be deleted from the graph in part (b) to make it into an Eulerian graph? Justify your answer.

**Solution:** There are four vertices of odd degree: $a, e, h, i$. If one edge of any graph is deleted, then the degrees of two vertices are decreased by one. Hence, to make the given graph into an Eulerian graph we need to delete at least two edges. Moreover, for the minimum number to be two, the two edges should have all four vertices $a, e, h, i$ as their ends. However, it is clearly not possible to chose two such edges among the three edges $\{e, a\}, \{e, h\}$ and $\{e, i\}$ with these ends. On the other hand, the graph becomes Eulerian if we delete the three edges $\{e, a\}, \{e, h\}$ and $\{e, i\}$ (or $\{f, a\}, \{e, h\}$ and $\{f, i\}$). Therefore, the minimum number is three.

(d) Show that the following graph is Hamiltonian by producing a spanning cycle.

![Graph](image)

**Solution:** A spanning cycle is $1, 5, 2, 7, 10, 11, 8, 3, 6, 9, 4, 1$.

(e) List all edges $e$ in the graph in part (d) such that the graph remains to be Hamiltonian after the edge $e$ is deleted. Explain why your list is complete.

**Solution:** The edges $\{1, 6\}, \{6, 11\}$ and $\{5, 10\}$ are not used in the spanning cycle produced in part (d). Hence, the deletion of any of these edges leaves a Hamiltonian graph. We will show that if delete any other edge then the graph becomes non-Hamiltonian. It is clear that if the deletion of an edge leaves a vertex of degree one, then the new graph is not Hamiltonian because it is not possible to include such a vertex in a spanning cycle. Hence it remains to show that the graphs obtained by deleting the edge $\{1, 5\}$ or $\{10, 11\}$ are not Hamiltonian. This holds since in both cases, if we also delete vertex 6, then each of the two new graphs will have two connected components. Thus, the complete list of edges is $\{1, 6\}, \{6, 11\}$ and $\{5, 10\}$.

\[1 + 1 + 3 + 2 + 3 = 10 \text{ marks}\]

5. (a) Complete the formulation of the Travelling Salesman Problem: “Given a connected weighted graph, find a walk which . . . ”.
Solution: Given a connected weighted graph, find a walk which visits every vertex, returns to its starting point, and has minimum weight subject to these two conditions.

(b) Find a walk which solves the Travelling Salesman Problem for the following weighted graph. Justify your answer.

\begin{center}
\begin{tikzpicture}[scale=0.6]
\node[vertex] (a) at (0,0) {a};
\node[vertex] (b) at (2,2) {b};
\node[vertex] (c) at (4,4) {c};
\node[vertex] (d) at (6,2) {d};
\node[vertex] (e) at (1,-2) {e};
\node[vertex] (f) at (3,-2) {f};
\draw[thick] (a) -- (b) node[midway, above] {12};
\draw[thick] (b) -- (c) node[midway, right] {25};
\draw[thick] (c) -- (d) node[midway, above] {14};
\draw[thick] (d) -- (e) node[midway, left] {25};
\draw[thick] (e) -- (f) node[midway, left] {12};
\draw[thick] (f) -- (a) node[midway, above] {24};
\end{tikzpicture}
\end{center}

Solution: The walk \(a, b, e, c, d, f, c, e, a\) of weight 113 is a solution. The distance from \(a\) to \(d\) is 49, the distance from \(d\) to \(f\) is 16 and the distance from \(f\) to \(a\) is 48. Hence the weight of a solution of the Travelling Salesman Problem cannot be less than \(49 + 16 + 48 = 113\).

(c) Amongst all trees with 10 vertices \(\{1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}\) consider those which have a vertex of degree 8.

(i) What is the possible number of leaves in such a tree?

Solution: Let \((d_1, \ldots, d_{10})\) be the degree sequence of such a tree. Since it has 9 edges we have \(d_1 + \cdots + d_{10} = 18\). We must have \(d_{10} = 8\) so that \(d_1 + \cdots + d_9 = 10\). Since all degrees are positive integers, we have \(d_1 = \cdots = d_8 = 1\) and \(d_9 = 2\). Hence the only possible number of leaves is 8.

(ii) What is the number of such trees?

Solution: Possible Prüfer sequences of such trees have the form \((a, \ldots, a, b, a, \ldots, a)\), where \(a\) and \(b\) are different elements of the set \(\{1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}\) and \(a\) occurs 7 times. The number of ways to choose a place for \(b\) is 8 so that the number of trees is \(8 \cdot 10 \cdot 9 = 720\).

(iii) What is the number of isomorphism classes of such trees?

Solution: There is one isomorphism class. Any such tree is obtained from the only nine-vertex tree \(T\) with eight leaves by adding another leaf adjacent to one of the leaves in \(T\).

Justify your answer to each of the questions.

(d) What is the number of spanning trees of the complete bipartite graph \(K_{2,4}\)? Justify your answer.

Solution: Let \(1, 2, 3, 4\) be the vertices of degree two in \(K_{2,4}\), and \(5, 6\) be the vertices of degree four. We need to find the number of ways to delete three edges from \(K_{2,4}\) so that the new graph is a tree. Clearly, for each \(v \in \{1, 2, 3, 4\}\) at most one edge with the end \(v\) can be deleted. There are 4 ways to choose three vertices in \(\{1, 2, 3, 4\}\) whose degrees will be decreased by one, and there are two ways to delete one of the edges adjacent to each vertex. Hence, the answer is \(4 \cdot 2 \cdot 2 = 32\).
6. This question is for MATH2069 students only.

(a) Let \( t \) be a natural number. Define what is meant for a graph \( G \) to be \( t \)-colourable.

**Solution:** A graph \( G \) is \( t \)-colourable if there exists a vertex colouring of \( G \) with \( t \) colours so that no two adjacent vertices have the same colour.

(b) Use mathematical induction on the number \( n \) of vertices in \( G \) to show that any graph \( G \) is \( t \)-colourable for \( t = \Delta(G) + 1 \).

**Solution:** In the case where \( G \) is a single vertex we have \( \Delta(G) = 0 \), and \( G \) is indeed 1-colourable. So we can assume that \( n \geq 2 \) and that the result is true for graphs with fewer than \( n \) vertices. Let \( v \) be any vertex of \( G \). It is clear that \( \Delta(G - v) \leq \Delta(G) \), so the induction hypothesis implies that \( G - v \) is \( (\Delta(G) + 1) \)-colourable; choose a vertex colouring of \( G - v \) where the colour set has size \( \Delta(G) + 1 \). Now the number of vertices of \( G \) adjacent to \( v \) is \( \deg(v) \), which is at most \( \Delta(G) \) by definition. So at most \( \Delta(G) \) different colours are used among the vertices adjacent to \( v \). Therefore there is a colour which is not used among these vertices, and we can colour \( v \) with this colour to obtain a vertex colouring of \( G \). The induction step is complete.

(c) The graph \( H \) is given by the picture

(i) What is the maximum possible value of the chromatic number \( \chi(H) \) as provided by the Brooks Theorem?

**Solution:** Since the graph is neither complete nor an odd cycle, its chromatic number \( \chi(H) \) does not exceed \( \Delta(H) = 4 \).

(ii) What is the exact value of \( \chi(H) \)? Justify your answer.

**Solution:** Since the graph contains 3-cycles, its chromatic number is at least 3. On the other hand, 3 colours are sufficient to get a vertex colouring: assign one colour to each pair of opposite vertices. Hence, \( \chi(H) = 3 \).

(iii) What are the possible values of the edge chromatic number \( \chi'(H) \) as provided by the Vizing Theorem?

**Solution:** Since \( \Delta(H) = 4 \), the possible values of the edge chromatic number \( \chi'(H) \) are 4 and 5.

(iv) What is the exact value of \( \chi'(H) \)? Justify your answer.

**Solution:** We will show that \( \chi'(H) = 4 \) by producing an edge colouring with 4 colours. Give the respective labels 1, 2, 3, 4, 5, 6 to the vertices starting with the top vertex 1 and proceeding clockwise.
6. This question is for MATH2969 students only.

(a) For any $n \geq 6$ the graph $G_n$ can be drawn as a regular polygon with $n$ vertices together with all diagonals of the minimal length.

[For instance, the picture above on this page represents $G_n$ for $n = 6$.]

(i) Calculate the edge chromatic number $\chi(G_n)$. Justify your answer.

**Solution:** Since the graph contains 3-cycles, its chromatic number is at least 3. On the other hand, by the Brooks theorem, $\chi(G_n)$ does not exceed $\Delta(G_n) = 4$. We will show that $\chi(G_n) = 3$ if $n$ is a multiple of 3, and $\chi(G_n) = 4$ otherwise.

Give the respective labels 1, 2, \ldots, $n$ to the vertices starting with a certain vertex 1 and proceeding clockwise. Suppose that $n = 3k$ for an integer $k \geq 2$. Colour the vertices 1, 4, \ldots, $3k - 2$ red, the vertices 2, 5, \ldots, $3k - 1$ blue and the vertices 3, 6, \ldots, $3k$ white. This yields a proper vertex colouring so that $\chi(G_{3k}) = 3$.

Now let $n = 3k + 1$ or $n = 3k + 2$ for an integer $k \geq 2$. We suppose that $\chi(G_n) = 3$ and will come to a contradiction. The 3-cycle 1, 2, 3 requires 3 colours so choose red for vertex 1, blue for vertex 2 and white for vertex 3. Since we cannot use any extra colours, the colours of all remaining vertices are then uniquely determined: we must colour vertex 4 red, vertex 5 blue, vertex 6 white, etc. Thus, the vertices of the form $3m + 1$, $3m + 2$ and $3m + 3$ with $m \geq 0$ must have respective colours red, blue and white. This makes a contradiction since one of the vertices $n - 1$ or $n - 2$ adjacent to vertex 1 will have the same colour as 1.

(ii) Calculate the edge chromatic number $\chi'(G_n)$. Justify your answer.

**Solution:** Since $\Delta(G_8) = 4$, by the Vizing Theorem the possible values of the edge chromatic number $\chi'(G_8)$ are 4 and 5. We will show that $\chi'(G_8) = 4$ by producing an edge colouring with 4 colours.

Give the respective labels 1, 2, 3, 4, 5, 6, 7, 8 as in the solution of the previous part. Colour the edges $\{1, 2\}$, $\{3, 4\}$, $\{5, 6\}$, $\{7, 8\}$ red, the edges $\{1, 3\}$, $\{2, 8\}$, $\{4, 6\}$, $\{5, 7\}$ blue, the edges $\{1, 7\}$, $\{2, 4\}$, $\{3, 5\}$, $\{6, 8\}$ yellow, and the edges $\{1, 8\}$, $\{2, 3\}$, $\{4, 5\}$, $\{6, 7\}$ white.

(iii) Prove that if $n$ is odd then $\chi'(G_n) = 5$.

**Solution:** By the Vizing Theorem, the possible values of the edge chromatic number $\chi'(G_n)$ are 4 and 5. We will show that an edge colouring with 4 colours is impossible. By the Hand-shaking Lemma, the number of edges is $2n$. On the other hand, the maximum number of edges with the same colour is $(n - 1)/2$. Hence, if only four different
colours are used, then the number of coloured edges would not exceed 
\[ 4 \cdot \frac{(n - 1)}{2} = 2n - 2. \]
Therefore, 4 colours are not sufficient and so 
\[ \chi'(G_n) = 5. \]

(b) Find the chromatic polynomial for the graph

\[ \text{Solution:} \] There is only one pair of non-adjacent vertices in the given graph 
\( G \), which we denote by \( v \) and \( w \). The graph \( G + \{v, w\} \) is isomorphic to the complete graph \( K_6 \) while the graph \( G[v, w] \) is isomorphic to \( K_5 \). Hence, by a theorem from lectures,

\[
P_G(t) = P_{K_6}(t) + P_{K_5}(t) = t(t - 1)(t - 2)(t - 3)(t - 4)(t - 5) \\
+ t(t - 1)(t - 2)(t - 3)(t - 4) = t(t - 1)(t - 2)(t - 3)(t - 4)^2.
\]

(3 + 2 + 3 + 2 = 10 marks)