MATH2069/2969: Discrete Mathematics and Graph Theory

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Time allowed: 2 hours, plus 10 minutes reading time

This booklet contains 5 pages.

This paper comprises 6 questions of equal value. Each question is divided into several parts.

Questions 1, 2, 4, 5 are the same for MATH2069 and MATH2969. For questions 3, 6 this paper contains both the normal-level MATH2069 question and the (completely different) advanced-level MATH2969 question. You may ONLY answer the questions for the unit you are enrolled in.

If you can’t solve one part of a question, you can still assume the result in doing later parts.

No notes or books are allowed. Approved calculators are permitted.
1. Let $A$ and $B$ be finite sets such that $|A| = r$ and $|B| = k$.
   (a) Give a formula for the number of functions $f : A \rightarrow B$. Explain briefly how this formula is derived.
   (b) Give a formula for the number of injective functions $f : A \rightarrow B$. Explain why this number is divisible by $r!$.
   (c) Suppose that $|A| = k + 1$ and $|B| = k$ for a positive integer $k$. Calculate the number of surjective functions $f : A \rightarrow B$. Give the explicit numerical value for $k = 7$.
   (d) Suppose that $|A| = r$ and $|B| = 2$, where $r \geq 2$. Find all values of $r$ for which the number of surjective functions from $A$ to $B$ coincides with the number of injective functions from $B$ to $A$. Give a proof that your list is complete.

2. (a) Write down the general solution of the recurrence relation
   $$b_n = -10b_{n-1} - 25b_{n-2} \text{ where } n \geq 2.$$  
   (b) Find a particular solution of the nonhomogeneous recurrence relation
   $$a_n = -10a_{n-1} - 25a_{n-2} + n \text{ where } n \geq 2.$$  
   (c) Find the general solution of the recurrence relation in part (b).
   (d) Find the general solution of the recurrence relation
   $$x_n = x_{n-1} + x_{n-2} \text{ where } n \geq 2.$$  
   (e) Recall that the Fibonacci and Lucas sequences $F_n$ and $L_n$ are $(0, 1, 1, 2, \ldots)$ and $(2, 1, 3, 4, \ldots)$, respectively. Given that $x_0 = x_1$ for the sequence $x_n$ in part (d), show that the ratio $x_n/(F_n + L_n)$ does not depend on $n$. 

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3. This question is for MATH2069 students only.

(a) Given a formal power series

$$A(z) = \sum_{n=0}^{\infty} a_n z^n$$

express the series

$$\sum_{n=0}^{\infty} (a_0 + a_1 + \cdots + a_n) z^n$$

in terms of $A(z)$. Justify your answer.

(b) Write a closed formula for the generating function of the sequence $a_n = (2n + 3) (-1)^n$.

(c) Calculate the sum $a_0 + a_1 + \cdots + a_n$, where $a_n$ is defined in part (b).

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3. This question is for MATH2969 students only.

Consider the Fibonacci and Lucas sequences as recalled in Question 2(e). The generating function for the Fibonacci numbers is

$$F(z) = \frac{z}{1 - z - z^2}.$$  

(a) Derive a closed formula for $L(z)$.

(b) Hence, or otherwise, prove the relation

$$F_n = \frac{L_{n-1}}{2} + \frac{L_{n-2}}{2^2} + \cdots + \frac{L_0}{2^n}$$

for all $n \geq 1$.

(c) Find an expression for $L_n$ as a linear combination of the Fibonacci numbers.
4.  (a) Consider the sequence of integers \((1, 1, 1, 1, 3, 3, k)\) with \(k \geq 3\).

   (i) Show that if \(k = 4\) then sequence is graphic. Draw a graph corresponding to this sequence.

   (ii) Show that if \(k \neq 4\) then the sequence is not graphic.

(b) A map of a region contains 18 cities labelled by 1, 2, \ldots, 18. For each \(k = 1, 2, \ldots, 9\) there is a road connecting city \(2k - 1\) and city \(2k\). Moreover, there is a road between any two even cities and there is a road between any two odd cities.

   (i) By interpreting the map as a graph, calculate the degrees of all its vertices.

   (ii) State the theorems from graph theory which imply that it is impossible to depart from city \(i\), use every road exactly once and arrive to city \(j\). Consider two cases: \(i = j\) and \(i \neq j\).

   (iii) State and apply another theorem from graph theory to show that it is possible to depart from city 1, visit every other city exactly once and return to city 1.

   (iv) Suppose the seven roads connecting city 1 with cities 3, 5, \ldots, 15 become unusable. Is it still possible to depart from city 1, visit every other city exactly once and return to city 1? Justify your answer.

5.  (a) Apply the Matrix–Tree theorem to count the number of spanning trees of the graph

\[
\begin{array}{c}
1 \\
3 \\
4 \\
2
\end{array}
\]

(b) Give a direct argument to count the number of spanning trees in part (a).

(c) Consider the trees with 7 vertices \(\{1, 2, 3, 4, 5, 6, 7\}\) which have a vertex of degree 4.

   (i) What is the possible number of leaves in such a tree?

   (ii) What is the number of isomorphism classes of such trees?

   Justify your answer to each of the questions.

(d) A tree has two vertices of degree 4. What is the minimum possible number of vertices in such a tree? Justify your answer.
6. This question is for MATH2069 students only.
   (a) Give the definition of the chromatic number of a graph \( G \).
   (b) Give the definition of the edge chromatic number of a graph \( G \).
   (c) The graph \( G \) has the form of a regular polygon with 10 vertices \( 1, 2, \ldots, 10 \) numbered clockwise, where, in addition, any two even vertices are connected by an edge.
      (i) State Brooks’ theorem. What does it say for the graph \( G \)?
      (ii) Find the chromatic number of \( G \).
      (iii) State Vizing’s theorem. What does it say for the graph \( G \)?
      (iv) Find the edge chromatic number of \( G \).

6. This question is for MATH2969 students only.
   For any \( n \geq 2 \) the graph \( G_n \) can be drawn as a regular polygon with \( 2n \) vertices \( 1, 2, \ldots, 2n \) numbered clockwise, where, in addition, any two even vertices are connected by an edge.
   (a) Find the chromatic number of \( G_n \).
   (b) Find the edge chromatic number of \( G_n \).
   (c) Give the definition of the chromatic polynomial of a graph \( G \).
   (d) Find the chromatic polynomial of the graph \( G_n \).

This is the end of the examination paper