This paper comprises 6 questions of equal value. Each question is divided into several parts.

Questions 1, 2, 4, 5 are the same for MATH2069 and MATH2969. For questions 3, 6 this paper contains both the normal-level MATH2069 question and the (completely different) advanced-level MATH2969 question. You may ONLY answer the questions for the unit you are enrolled in.

If you can’t solve one part of a question, you can still assume the result in doing later parts.

No notes or books are allowed. Approved calculators are permitted.
1. Let $A$ and $B$ be finite sets such that $|A| = r$ and $|B| = k$.

(a) Give a formula for the number of functions $f : A \to B$. Explain briefly how this formula is derived.

Solution: The number is $k^r$. For each element $x \in A$ there are $k$ choices for the image $f(x) \in B$. By the multiplication principle, the number of functions is the product $k \times k \times \cdots \times k$ ($r$ factors).

(b) Give a formula for the number of injective functions $f : A \to B$. Explain why this number is divisible by $r!$.

Solution: The number is $k^{(r)}$. The ratio $\frac{k^{(r)}}{r!}$ equals the binomial coefficient $\binom{k}{r}$ and hence, an integer.

(c) Suppose that $|A| = k + 1$ and $|B| = k$ for a positive integer $k$. Calculate the number of surjective functions $f : A \to B$. Give the explicit numerical value for $k = 7$.

Solution: The number is $k! S(k+1,k)$. This also equals $k! \left(\begin{array}{c} k+1 \\ 2 \end{array}\right)$. For $k = 7$ this equals $7! \left(\begin{array}{c} 8 \\ 2 \end{array}\right) = 5040 \times 28 = 141120$.

(d) Suppose that $|A| = r$ and $|B| = 2$, where $r \geq 2$. Find all values of $r$ for which the number of surjective functions from $A$ to $B$ coincides with the number of injective functions from $B$ to $A$. Give a proof that your list is complete.

Solution: The number of surjective functions from $A$ to $B$ is $2! S(r, 2) = 2^r - 2$. The number of injective functions from $B$ to $A$ is $r(2) = r(r-1) = r^2 - r$. The equality $2^r - 2 = r^2 - r$ holds for $r = 2$ and $r = 3$. To prove that this list is complete, we show by induction that $2^r - 2 > r^2 - r$ for all $r \geq 4$. This is true for $r = 4$ since $2^4 - 2 = 14 > 12 = 4^2 - 4$. Now suppose that $r \geq 5$ and that the inequality holds for $r - 1$. Then by the induction hypothesis,

$$2^r - 2 = 2(2^{r-1}) - 2 > 2((r - 1)^2 - (r - 1) + 2) - 2 = 2r^2 - 6r + 6.$$ 

It remains to note that $2r^2 - 6r + 6 > r^2 - r$ since $r^2 - 5r + 6 = (r-2)(r-3) > 0$ for $r \geq 5$.

2. (a) Write down the general solution of the recurrence relation

$$b_n = -10 b_{n-1} - 25 b_{n-2} \text{ where } n \geq 2.$$ 

Solution: The characteristic polynomial is $x^2 + 10x + 25 = (x+5)^2$. Hence the general solution of the recurrence relation is $b_n = (An + B)(-5)^n$, where $A$ and $B$ are arbitrary constants.

(b) Find a particular solution of the nonhomogeneous recurrence relation

$$a_n = -10 a_{n-1} - 25 a_{n-2} + n \text{ where } n \geq 2.$$ 

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Solution: We will look for a particular solution in the form \( p_n = Cn + D \).
Substitute into the relation,

\[
Cn + D = -10(C(n - 1) + D) - 25(C(n - 2) + D) + n.
\]

Equating the coefficients of \( n \) and the constant terms on both sides, we find \( C = -10C - 25C + 1 \) and \( D = -10(D - C) - 25(D - 2C) \). Hence, \( C = \frac{1}{36} \) and \( D = \frac{60}{36^2} = \frac{5}{108} \). Thus, \( p_n = \frac{n}{36} + \frac{5}{108} \).

(c) Find the general solution of the recurrence relation in part (b).

Solution: By a result from lectures, the general solution is

\[
a_n = (An + B)(-5)^n + \frac{n}{36} + \frac{5}{108}.
\]

(d) Find the general solution of the recurrence relation \( x_n = x_{n-1} + x_{n-2} \) where \( n \geq 2 \).

Solution: The characteristic polynomial is \( x^2 - x - 1 \). Its roots are \( \frac{1 \pm \sqrt{5}}{2} \).

Hence, the general solution is

\[
x_n = A \left( \frac{1 + \sqrt{5}}{2} \right)^n + B \left( \frac{1 - \sqrt{5}}{2} \right)^n,
\]

where \( A \) and \( B \) are arbitrary constants.

(e) Recall that the Fibonacci and Lucas sequences \( F_n \) and \( L_n \) are \( (0, 1, 1, 2, \ldots) \) and \( (2, 1, 3, 4, \ldots) \), respectively. Given that \( x_0 = x_1 \) for the sequence \( x_n \) in part (d), show that the ratio \( x_n/(F_n + L_n) \) does not depend on \( n \).

Solution: The Fibonacci and Lucas sequences \( F_n \) and \( L_n \) satisfy the recurrence relation of part (d). Since these sequences are not proportional to each other, the sequence \( x_n \) must be their linear combination; that is, \( x_n = CF_n + DL_n \) for some constants \( C \) and \( D \). The condition \( x_0 = x_1 \) gives \( 2D = C + D \) and so, \( C = D \). Hence, the ratio \( x_n/(F_n + L_n) \) equals \( C \), and does not depend on \( n \).
3. This question is for MATH2069 students only.

(a) Given a formal power series

$$A(z) = \sum_{n=0}^{\infty} a_n z^n$$

express the series

$$\sum_{n=0}^{\infty} (a_0 + a_1 + \cdots + a_n) z^n$$

in terms of $A(z)$. Justify your answer.

**Solution:** We have

$$\sum_{n=0}^{\infty} (a_0 + a_1 + \cdots + a_n) z^n = A(z) \frac{1}{1-z}.$$ 

This follows by using the definition of the product of formal power series,

$$A(z) \frac{1}{1-z} = \left(\sum_{n=0}^{\infty} a_n z^n\right) \left(\sum_{n=0}^{\infty} z^n\right) = \sum_{n=0}^{\infty} (a_0 + a_1 + \cdots + a_n) z^n.$$

(b) Write a closed formula for the generating function of the sequence $a_n = (2n + 3)(-1)^n$.

**Solution:** By a theorem from lectures,

$$\sum_{n=0}^{\infty} (2n + 3)(-1)^n z^n = \sum_{n=0}^{\infty} (2n + 3) (-z)^n = -\frac{2z}{(1+z)^2} + \frac{3}{1+z}.$$ 

(c) Calculate the sum $a_0 + a_1 + \cdots + a_n$, where $a_n$ is defined in part (b).

**Solution:** By part (a), the generating function for the sums is

$$-\frac{2z}{(1-z)(1+z)^2} + \frac{3}{(1-z)(1+z)} = \frac{3+z}{(1-z)(1+z)^2}.$$ 

Use partial fraction decomposition

$$\frac{3+z}{(1-z)(1+z)^2} = \frac{A}{1-z} + \frac{B}{1+z} + \frac{C}{(1+z)^2},$$

so that the constants $A$, $B$ and $C$ are found from the identity

$$3+z = A(1+z)^2 + B(1+z)(1-z) + C(1-z).$$

Both sides are polynomials in $z$, so that taking $z = 1$ we find $A = 1$, and by taking $z = -1$ we get $C = 1$. Furthermore, by taking $z = 0$ we get $B = 1$. Therefore, the generating function for the sums equals

$$\frac{1}{1-z} + \frac{1}{1+z} + \frac{1}{(1+z)^2} = \sum_{n=0}^{\infty} z^n + \sum_{n=0}^{\infty} (-z)^n + \sum_{n=0}^{\infty} (n+1)(-z)^n$$

and so $a_0 + a_1 + \cdots + a_n = 1 + (-1)^n + (n+1)(-1)^n = (n+2)(-1)^n + 1.$
3. This question is for MATH2969 students only.
Consider the Fibonacci and Lucas sequences as recalled in Question 2(e). The generating function for the Fibonacci numbers is
\[ F(z) = \frac{z}{1 - z - z^2}. \]

(a) Derive a closed formula for \( L(z) \).
Solution: The recurrence relation is \( L_n = L_{n-1} + L_{n-2} \) for \( n \geq 2 \) with \( L_0 = 2 \) and \( L_1 = 1 \). Hence,
\[ L(z) = 2 + z + \sum_{n=2}^{\infty} (L_{n-1} + L_{n-2}) z^n = 2 + z + (L(z) - 2) + z^2 L(z), \]
which gives
\[ L(z) = \frac{2 - z}{1 - z - z^2}. \]

(b) Hence, or otherwise, prove the relation
\[ F_n = \frac{L_{n-1}}{2} + \frac{L_{n-2}}{2^2} + \cdots + \frac{L_0}{2^n} \]
for all \( n \geq 1 \).
Solution: We have
\[ \frac{F(z)}{L(z)} = \frac{z}{2 - z} = \sum_{n=1}^{\infty} \frac{z^n}{2^n}. \]
Hence,
\[ F(z) = L(z) \sum_{n=1}^{\infty} \frac{z^n}{2^n}. \]
Comparing the coefficients of \( z^n \) on both sides we get
\[ F_n = \sum_{m=1}^{n} \frac{L_{n-m}}{2^m}, \]
as required.
An alternative solution is proof by induction on \( n \).

(c) Find an expression for \( L_n \) as a linear combination of the Fibonacci numbers.
Solution: We have
\[ \frac{L(z)}{F(z)} = \frac{2 - z}{z} \]
and so,
\[ L(z) = 2 \frac{F(z)}{z} - F(z). \]
Comparing the coefficients of \( z^n \) on both sides we get
\[ L_n = 2 F_{n+1} - F_n \]
for \( n \geq 0 \).
An alternative solution is to derive this from part (b).
4. (a) Consider the sequence of integers $(1, 1, 1, 1, 3, 3, k)$ with $k \geq 3$.

(i) Show that if $k = 4$ then sequence is graphic. Draw a graph corresponding to this sequence.

**Solution:** By a theorem from lectures, the sequence $(1, 1, 1, 1, 3, 3, 4)$ is graphic if and only if $(0, 0, 1, 1, 2, 2)$ is graphic. The latter holds if and only if $(0, 0, 0, 1, 1)$ is graphic, which is the case. Therefore, we can conclude that the sequence $(1, 1, 1, 1, 3, 3, 4)$ is graphic. A possible graph is

(ii) Show that if $k \neq 4$ then the sequence is not graphic.

**Solution:** First observe that for the sequence to be graphic we must have $k \leq 6$ since the number of vertices of the graph should be 7. On the other hand, by the Hand-shaking Lemma, the sum of all degrees should be an even number. This rules out the values $k = 3$ and $k = 5$. The only remaining value is $k = 6$. The sequence $(1, 1, 1, 1, 3, 3, 6)$ is graphic if and only if $(0, 0, 0, 0, 2, 2)$ is graphic. However, $(0, 0, 0, 0, 2, 2)$ is not graphic since the number of vertices of positive degrees would be 2 and so it is impossible for a vertex to have degree 2. Thus, $(1, 1, 1, 1, 3, 3, 6)$ is not graphic.

(b) A map of a region contains 18 cities labelled by $1, 2, \ldots, 18$. For each $k = 1, 2, \ldots, 9$ there is a road connecting city $2k - 1$ and city $2k$. Moreover, there is a road between any two even cities and there is a road between any two odd cities.

(i) By interpreting the map as a graph, calculate the degrees of all its vertices.

**Solution:** Since the total number of odd vertices is 9, each odd vertex is adjacent to 8 odd vertices and one even vertex. Hence, the degree of each odd vertex is 9. Similarly, the degree of each even vertex is also 9.

(ii) State the theorems from graph theory which imply that it is impossible to depart from city $i$, use every road exactly once and arrive to city $j$. Consider two cases: $i = j$ and $i \neq j$.

**Solution:** Theorem: A connected graph $G$ is Eulerian if and only if $\deg(v)$ is even for every vertex $v$ of $G$. Since all degrees of our graph are odd, this theorem implies that the graph does not contain an Eulerian circuit and so, it is impossible to depart from city $i$, use every road exactly once and return to city $i$. Another theorem: A connected graph $G$ has an Eulerian trail if and only if it has exactly two vertices of odd degree. This theorem implies that it is impossible to depart from city $i$, use every road exactly once and arrive to city $j$, where $i \neq j$. 

SOLUTIONS
(iii) State and apply another theorem from graph theory to show that it is possible to depart from city 1, visit every other city exactly once and return to city 1.

**Solution:** Ore’s Theorem: Let $G$ be a connected graph with $n \geq 3$ vertices. If every non-adjacent pair of vertices $v, w$ satisfies $\deg(v) + \deg(w) \geq n$, then $G$ is Hamiltonian. Our graph is regular of degree 9 and the number of vertices is 18. Hence, the conditions of Ore’s Theorem are satisfied and we may conclude that the graph is Hamiltonian. This means that it is possible to depart from city 1, visit every other city exactly once and return to city 1.

(iv) Suppose the seven roads connecting city 1 with cities 3, 5, …, 15 become unusable. Is it still possible to depart from city 1, visit every other city exactly once and return to city 1? Justify your answer.

**Solution:** It is still possible. One solution is given by the spanning cycle 1, 2, 6, 8, 10, 12, 14, 16, 18, 4, 3, 5, 7, 9, 11, 13, 15, 17, 1.

5. (a) Apply the Matrix–Tree theorem to count the number of spanning trees of the graph

![Graph Diagram]

**Solution:** Kirchoff’s Matrix–Tree theorem: Let $M$ be the Laplacian matrix of a graph $G$ with vertex set $\{1, \ldots, n\}$ with $n \geq 2$. Then for all $k, l \in \{1, \ldots, n\}$, the $(k, l)$-cofactor $(-1)^{k+l} \det M^{kl}$ equals the number of spanning trees of $G$.

The Laplacian matrix of this graph is

$$M = \begin{pmatrix} 2 & -1 & -1 & 0 \\ -1 & 3 & -1 & -1 \\ -1 & -1 & 3 & -1 \\ 0 & -1 & -1 & 2 \end{pmatrix},$$

and the number of spanning trees equals any cofactor of $M$. The $(3,3)$-cofactor is:

$$\det \begin{pmatrix} 2 & -1 & 0 \\ -1 & 3 & -1 \\ 0 & -1 & 2 \end{pmatrix}$$

which equals

$$2 \cdot 3 \cdot 2 - 2 - 2 = 8.$$

(b) Give a direct argument to count the number of spanning trees in part (a).

**Solution:** The number of edges in a spanning tree must be 3 so that we need to delete two edges from the graph. First, if delete the edge $\{2, 3\}$, then
there will be 4 ways to delete another edge. If the edge \( \{2, 3\} \) is not deleted, then there are 2 ways to delete one edge in the pair \( \{1, 2\}, \{1, 3\} \) and 2 ways to delete one edge in the pair \( \{2, 4\}, \{3, 4\} \). The total number is \( 4 + 2 \cdot 2 = 8 \).

(c) Consider the trees with 7 vertices \( \{1, 2, 3, 4, 5, 6, 7\} \) which have a vertex of degree 4.

(i) What is the possible number of leaves in such a tree?

**Solution:** Let \((d_1, \ldots, d_7)\) be the degree sequence of such a tree. Since it has 6 edges we have \(d_1 + \cdots + d_7 = 12\). We must have \(d_7 = 4\) so that \(d_1 + \cdots + d_6 = 8\). Since all degrees are positive integers, we have only two possibilities \(d_1 = \cdots = d_5 = 1\) and \(d_6 = 3\), or \(d_1 = \cdots = d_4 = 1\) and \(d_5 = d_6 = 2\). Such trees do exist: for example, take Prüfer sequences \((1, 1, 2, 2, 2)\) and \((1, 2, 3, 3, 3)\). Hence the possible number of leaves is 4 or 5.

(ii) What is the number of isomorphism classes of such trees?

**Solution:** There are three isomorphism classes. Label the vertex of degree 4 by \(a\). It must be adjacent to 4 other vertices which we denote by \(b, c, d\) and \(e\). There are two more vertices, say, \(f\) and \(g\). Since the graph is connected, \(f\) or \(g\) should be adjacent to one of the 4 vertices \(b, c, d\) or \(e\). Re-labelling vertices if necessary, we may assume that \(f\) is adjacent to \(b\). Now, there are 3 choices (up to isomorphism): \(g\) can be adjacent to \(b, c\) or \(f\). The three trees are pairwise non-isomorphic: the first contains a vertex of degree 3, the third has three paths of length 4.

Justify your answer to each of the questions.

(d) A tree has two vertices of degree 4. What is the minimum possible number of vertices in such a tree? Justify your answer.

**Solution:** Let the two vertices of degree 4 be 1 and 2. Then each of 1 and 2 will occur in the Prüfer sequence of such a tree three times. Hence, the total number of vertices cannot be less than 8. Such trees with 8 vertices do exist: take vertices 1, 2, 3, 4, 5, 6, 7, 8 and the Prüfer sequence \((1, 1, 1, 2, 2, 2)\).
6. This question is for MATH2069 students only.
(a) Give the definition of the chromatic number of a graph $G$.

Solution: The chromatic number of $G$ is the minimum number of colours required in a vertex colouring, so that no two adjacent vertices are assigned the same colour.

(b) Give the definition of the edge chromatic number of a graph $G$.

Solution: The edge chromatic number of $G$ is the minimum number of colours required in an edge colouring, so that no two edges with a common end are assigned the same colour.

(c) The graph $G$ has the form of a regular polygon with 10 vertices $1, 2, \ldots, 10$ numbered clockwise, where, in addition, any two even vertices are connected by an edge.

(i) State Brooks’ theorem. What does it say for the graph $G$?

Solution: Brooks’ theorem: Let $G$ be a connected graph which is neither complete nor an odd cycle. Then the chromatic number of $G$ does not exceed the maximal vertex degree $\Delta(G)$.

The given graph is neither complete nor an odd cycle. Hence its chromatic number does not exceed the maximal degree, which is 6.

(ii) Find the chromatic number of $G$.

Solution: Since $G$ contains a subgraph isomorphic to $K_5$, the chromatic number is at least 5. In fact, the number is 5. Choose a colour for each even vertex. Then for each odd vertex use the colour of the opposite even vertex of the polygon.

(iii) State Vizing’s theorem. What does it say for the graph $G$?

Solution: Vizing’s theorem: The edge-chromatic number of any graph $G$ is either $\Delta(G)$ or $\Delta(G) + 1$.

The edge-chromatic number of the given graph is either 6 or 7.

(iv) Find the edge chromatic number of $G$.

Solution: We will show that the edge-chromatic number of the graph is 6 by providing an edge colouring with 6 colours. First consider the subgraph of $G$ isomorphic to $K_5$ with the vertices $2, 4, 6, 8, 10$. For each of these vertices choose a colour and use it for the opposite side of the pentagon and for the diagonal parallel to that side. Furthermore, use the same colour for the edge connecting that vertex and the next odd vertex clockwise in $G$. We have thus used 5 colours. Now use the 6-th colour for the remaining 5 edges.

6. This question is for MATH2969 students only.

For any $n \geq 2$ the graph $G_n$ can be drawn as a regular polygon with $2n$ vertices $1, 2, \ldots, 2n$ numbered clockwise, where, in addition, any two even vertices are connected by an edge.

SOLUTIONS
(a) Find the chromatic number of $G_n$.

**Solution:** We will show that $\chi(G_n) = n$ for $n \geq 3$ and $\chi(G_2) = 3$. For $G_2$ use three colours for 1, 2, 4, and the same colour for 1 and 3. Now suppose that $n \geq 3$. The graph $G_n$ contains a subgraph isomorphic to $K_n$: its vertices are $2, 4, \ldots, 2n$. Choose $n$ colours for the vertices of this subgraph. For each odd vertex of $G$ use the same colour as for the opposite vertex of the polygon.

(b) Find the edge chromatic number of $G_n$.

**Solution:** Since $\Delta(G_n) = n + 1$, the Vizing theorem implies that $\chi'(G_n)$ is $n + 1$ or $n + 2$. We will show that $\chi'(G_n) = n + 1$ by producing an edge colouring with $n + 1$ colours. Suppose first that $n$ is odd. Consider the subgraph of $G$ isomorphic to $K_n$ with the vertices $2, 4, \ldots, 2n$. For each of these vertices choose a colour and use it for the opposite side of the pentagon and all the diagonals parallel to that side. Furthermore, use the same colour for the edge connecting that vertex and the next odd vertex clockwise in $G_n$. We have thus used $n$ colours. Now use the $(n + 1)$-th colour for the remaining $n$ edges.

Now let $n$ be even. We know that $\chi'(K_n) = n - 1$. So we can use $n - 1$ colours for the edges of the subgraph of $G_n$ isomorphic to $K_n$. Then use two extra colours for the remaining edges of the cycle $C_{2n}$.

(c) Give the definition of the chromatic polynomial of a graph $G$.

**Solution:** For a natural number $t$, the chromatic polynomial $P_G(t)$ is the number of vertex colourings of $G$ with a fixed colour set of size $t$.

(d) Find the chromatic polynomial of the graph $G_n$.

**Solution:** Consider the subgraph of $G$ isomorphic to $K_n$ with the vertices $2, 4, \ldots, 2n$. The number of vertex colourings of this subgraph is $t_{(n)} = t(t-1)\ldots(t-n+1)$. Given such a colouring, for each of the remaining $n$ odd vertices the number of available colours is $t - 2$. Hence, the chromatic polynomial of the graph $G_n$ is $t_{(n)}(t-2)^n$. 

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**SOLUTIONS** This is the end of the examination paper.