1. Prove by induction that, for all $n \geq 0$,
   (a) $n^3 + 5n$ is a multiple of 3 (i.e. $n^3 + 5n = 3\ell$ for some integer $\ell$).

   **Solution:** The $n = 0$ case holds because $0^3 + 0 = 0$ is a multiple of 3 (it is $3 \times 0$). Suppose that $n \geq 1$ and that the result is known for $n - 1$, i.e.
   
   $$(n - 1)^3 + 5(n - 1) = 3\ell,$$
   
   for some integer $\ell$.

   Then
   
   $$3\ell = n^3 - 3n^2 + 3n - 1 + 5n - 5 = n^3 + 5n - 3(n^2 - n + 2),$$
   
   so $n^3 + 5n = 3(\ell + n^2 - n + 2)$ is a multiple of 3, establishing the inductive step and completing the proof.

   (b) $5^n - 4n - 1$ is a multiple of 16.

   **Solution:** The $n = 0$ case holds because $5^0 - 4 \times 0 - 1 = 0$ is a multiple of 16. Suppose that $n \geq 1$ and that the result is known for $n - 1$, i.e.

   $$5^{n-1} - 4(n - 1) - 1 = 16\ell,$$ for some integer $\ell$.

   This equation can be rewritten as

   $$5^{n-1} = 4n - 3 + 16\ell.$$

   So

   $$5^n - 4n - 1 = 5(4n - 3 + 16\ell) - 4n - 1 = 16(n - 1 + 5\ell),$$

   which is a multiple of 16, establishing the inductive step and completing the proof.

2. Use the characteristic polynomial to solve the following recurrence relations:
   (a) $a_n = 5a_{n-1} - 6a_{n-2}$ for $n \geq 2$, where $a_0 = 2$, $a_1 = 5$.

   **Solution:** The characteristic polynomial is $x^2 - 5x + 6 = (x - 2)(x - 3)$ with roots 2 and 3, so the general solution is $a_n = C_12^n + C_23^n$ for some constants $C_1, C_2$. In our case we have

   $$2 = a_0 = C_1 + C_2 \quad \text{and} \quad 5 = a_1 = 2C_1 + 3C_2.$$

   Solving yields $C_1 = C_2 = 1$, so the solution is

   $$a_n = 2^n + 3^n.$$
(b) \( a_n = 4a_{n-1} - 3a_{n-2} \) for \( n \geq 2 \), where \( a_0 = -1 \), \( a_1 = 2 \).

**Solution:** The characteristic polynomial is \( x^2 - 4x + 3 = (x - 1)(x - 3) \) with roots 1 and 3, so the general solution is \( a_n = C_11^n + C_23^n \) for some constants \( C_1 \), \( C_2 \). In our case we have

\[-1 = a_0 = C_1 + C_2 \quad \text{and} \quad 2 = a_1 = C_1 + 3C_2.\]

Solving yields \( C_1 = -5/2 \) and \( C_2 = 3/2 \), so the solution is

\[a_n = \frac{3^{n+1} - 5}{2}.\]

(c) \( a_n = 4a_{n-1} - 4a_{n-2} \) for \( n \geq 2 \), where \( a_0 = 3 \), \( a_1 = 8 \).

**Solution:** The characteristic polynomial is \( x^2 - 4x + 4 = (x - 2)^2 \) with repeated root 2, so the general solution is \( a_n = C_12^n + C_2n2^n \) for some constants \( C_1 \), \( C_2 \). In our case we have

\[3 = a_0 = C_1 \quad \text{and} \quad 8 = a_1 = 2C_1 + 2C_2,\]

yielding \( C_1 = 3 \) and \( C_2 = 1 \), so the final solution is

\[a_n = 3 \times 2^n + n2^n = (n + 3)2^n.\]

(d) \( a_n = 6a_{n-1} - 9a_{n-2} \) for \( n \geq 2 \), where \( a_0 = 2 \), \( a_1 = -3 \).

**Solution:** The characteristic polynomial is \( x^2 - 6x + 9 = (x - 3)^2 \) with repeated root 3, so the general solution is \( a_n = C_13^n + C_2n3^n \) for some constants \( C_1 \), \( C_2 \). In our case we have

\[2 = a_0 = C_1 \quad \text{and} \quad -3 = a_1 = 3C_1 + 3C_2,\]

yielding \( C_1 = 2 \) and \( C_2 = -3 \), so that the final solution is

\[a_n = 3^n(2 - 3n).\]

*(e)* \( a_n = 6a_{n-1} - 11a_{n-2} + 6a_{n-3} \) for \( n \geq 3 \), where \( a_0 = 3 \), \( a_1 = 5 \), \( a_2 = 11 \).

**Solution:** The characteristic polynomial is \( x^3 - 6x^2 + 11x - 6 = (x - 1)(x - 2)(x - 3) \) with roots 1, 2, 3, so the general solution is

\[a_n = C_1 + C_22^n + C_33^n\]

for some constants \( C_1 \), \( C_2 \), \( C_3 \). In our case we have

\[3 = a_0 = C_1 + C_2 + C_3, \quad 5 = a_1 = C_1 + 2C_2 + 3C_3, \quad 11 = C_1 + 4C_2 + 9C_3,\]

yielding \( C_1 = 2 \), \( C_2 = 0 \) and \( C_3 = 1 \), so the final solution is

\[a_n = 3^n + 2.\]
\( a_n = 6a_{n-1} - 12a_{n-2} + 8a_{n-3} \) for \( n \geq 3 \), where \( a_0 = 2, a_1 = 4, a_2 = 16 \).

**Solution:** The characteristic polynomial is \( x^3 - 6x^2 + 12x - 8 = (x - 2)^3 \) with repeated root 2, so the general solution is

\[
a_n = C_12^n + C_2n2^n + C_3n^22^n
\]

for some constants \( C_1, C_2, C_3 \). In our case we have

\[
2 = a_0 = C_1, \quad 4 = a_1 = 2C_1 + 2C_2 + 2C_3, \quad 16 = a_2 = 4C_1 + 8C_2 + 16C_3,
\]

yielding \( C_1 = 2, C_2 = -1 \) and \( C_3 = 1 \), so the final solution is

\[
a_n = 2^n(2 - n + n^2).
\]

3. Companies A and B control the market for a certain product. From one year to the next, A retains 70% of its custom and loses to B the remaining 30%, while B retains 60% of its custom and loses to A the remaining 40%. Let \( a_n \) denote the market share of company A after \( n \) years (thus, that of company B is \( 1 - a_n \)).

(a) Write down a recurrence relation expressing \( a_n \) in terms of \( a_{n-1} \), for \( n \geq 1 \).

**Solution:** For \( n \geq 1 \), we have

\[
a_n = \frac{7}{10}a_{n-1} + \frac{4}{10}(1-a_{n-1}) = \frac{3}{10}a_{n-1} + \frac{4}{10}.
\]

(b) Solve the recurrence relation, in the sense of giving a closed formula for \( a_n \), in terms of \( a_0 \).

**Solution:** Unravelling the recurrence relation, we get:

\[
a_n = \frac{3}{10}a_{n-1} + \frac{4}{10} = \frac{3}{10} \left( \frac{3}{10}a_{n-2} + \frac{4}{10} \right) + \frac{4}{10}
\]

\[
= \left( \frac{3}{10} \right)^2 a_{n-2} + \frac{3}{10} \frac{4}{10} + \frac{4}{10} = \left( \frac{3}{10} \right)^2 \left( \frac{3}{10}a_{n-3} + \frac{4}{10} \right) + \frac{3}{10} \frac{4}{10} + \frac{4}{10}
\]

\[
= \left( \frac{3}{10} \right)^3 a_{n-3} + \left( \frac{3}{10} \right)^2 \frac{4}{10} + \frac{3}{10} \frac{4}{10} + \frac{4}{10}
\]

\[
\vdots
\]

\[
= \left( \frac{3}{10} \right)^n a_0 + \frac{4}{10} \left[ \left( \frac{3}{10} \right)^{n-1} + \ldots + \frac{3}{10} + 1 \right]
\]

\[
= \left( \frac{3}{10} \right)^n a_0 + \frac{4}{10} \left( \frac{3}{10} \right)^{n-1} - 1
\]

\[
= \left( a_0 - \frac{4}{7} \right) \left( \frac{3}{10} \right)^n + \frac{4}{7}.
\]

Here the second-last equality uses the formula for the sum of a geometric progression.
(c) Hence prove that the market share of company A in the long run (i.e. the limit of \(a_n\) as \(n \to \infty\)) is independent of its initial market share \(a_0\).

**Solution:** As \(n \to \infty\), the power \((\frac{3}{10})^n\) tends to 0, so \(a_n \to \frac{4}{7}\). So whatever the initial situation, the market tends to a stable situation where company A has a \(\frac{4}{7}\) market share and company B has a \(\frac{3}{7}\) market share.

4. Let \(b_n\) be the number of ways of forming a line of \(n\) people distinguished only by whether they are male (M) or female (F), such that no two males are next to each other. For example, the possibilities with 3 people are FFF, FFM, FMF, MFF, and MFM, so \(b_3 = 5\). Write down a recurrence relation for \(b_n\). Do you recognize the sequence?

**Solution:** We have \(b_0 = 1\), \(b_1 = 2\), and \(b_n = b_{n-1} + b_{n-2}\) if \(n \geq 2\). To see this notice that in a line of \(n\) people with \(n \geq 2\), if the last person is female then there are \(b_{n-1}\) possibilities for the line of the first \(n-1\) people, whilst if the last person is male then the second last person must be female, so that there are \(b_{n-2}\) possibilities for the line of the first \(n-2\) people. We get the Fibonacci sequence with the first two terms deleted, so \(b_n = F_n + 2\).

5. Define a sequence recursively by \(a_0 = 1\), \(a_1 = 2\), and \(a_n = a_{n-1}a_{n-2}\) for \(n \geq 2\).

(a) Find \(a_2\), \(a_3\), \(a_4\), \(a_5\) and \(a_6\).

**Solution:** \(a_2 = 2\), \(a_3 = 4 = 2^2\), \(a_4 = 8 = 2^3\), \(a_5 = 32 = 2^5\), \(a_6 = 2^8\).

(b) Prove that \(a_n = 2^{F_n}\), where \(F_0, F_1, F_2, \ldots\) is the Fibonacci sequence.

**Solution:** As seen in lectures, we only need to show that \(2^{F_n}\) satisfies the same initial conditions and recurrence relation as \(a_n\). The initial conditions hold because \(2^{F_0} = 2^0 = 1\) and \(2^{F_1} = 2^1 = 2\). The recurrence relation holds because for \(n \geq 2\),

\[
2^{F_n} = 2^{F_{n-1}+F_{n-2}} = 2^{F_{n-1}}2^{F_{n-2}},
\]

by the Fibonacci recurrence relation \(F_n = F_{n-1} + F_{n-2}\).

6. Imagine a \(2^n \times 2^n\) array of equal-sized squares, where \(n\) is some positive integer. We want to cover this array with non-overlapping L-shaped tiles, each of which exactly covers three squares (one square and two of the adjacent squares, not opposite to each other). Since the number of squares is not a multiple of 3, we need to remove one square before we start. Prove by induction that no matter which square we remove, the remaining squares can be covered by these L-shaped tiles.

**Solution:** The base case is clear, because removing a square from a \(2 \times 2\) array leaves 3 squares which can be covered by a single tile. We now prove the claim for \(n \geq 2\), assuming its truth for \(n-1\). Let \(G\) be the \(2^n \times 2^n\) array with exactly one square missing. Denote the quarters of \(G\) by \(UL\) for upper left, \(UR\) for upper right, \(LL\) for lower left and \(LR\) for lower right. Each quarter is a \(2^{n-1} \times 2^{n-1}\) array, except that one of the quarters has one square missing. By rotating \(G\) if necessary, we may suppose that the missing square is in \(LL\). Let \(T\) be the L-shape formed by the lower-rightmost subsquare of \(UL\), the lower-leftmost subsquare of \(UR\) and the
The upper-leftmost subsquare of $LR$. Then removing $T$ from $G$ produces a union of four $2^{n-1} \times 2^{n-1}$ arrays each with one square missing, and each of these can be tiled by L-shapes, by the induction hypothesis. Hence $G$ is tiled by all these L-shapes together with $T$, establishing the inductive step and completing the proof.

*7. The following argument ‘proves’ that whenever a group of people is in the same room, they all have the same height. There must be an invalid step; find it.

We argue by induction on the number $n$ of people in the room. The $n = 1$ case is obviously true. Suppose that $n \geq 2$ and that the claim holds for rooms with $n-1$ people. Let $P_1, P_2, \ldots, P_n$ be the $n$ people in this room. If $P_n$ were to leave the room we would have a room with $n-1$ people, so by the inductive hypothesis, $P_1, P_2, \ldots, P_{n-1}$ all have the same height. We can apply the same reasoning with $P_1$ leaving the room, so $P_2, \ldots, P_{n-1}, P_n$ all have the same height. But $P_2$ is in both these collections, so all of $P_1, P_2, \ldots, P_n$ have the same height. This establishes the inductive step, and so the claim holds for all $n$ by induction.

**Solution:** Since the claim is false even when $n = 2$, the proof must fail already in this case; when you run through the argument with $n = 2$, the error emerges. The invalid step is the assertion that “$P_2$ is in both these collections”, because this ignores the convention governing the way these collections were written out. When you start with $n$ people $P_1, P_2, \ldots, P_n$ and remove $P_n$, it is reasonable to list the remaining people as “$P_1, P_2, \ldots, P_{n-1}$”, but you have to bear in mind that if $n = 2$, this list will just consist of $P_1$ and will not in fact include $P_2$.

*8. For which $n$ is the Fibonacci number $F_n$ even, and for which $n$ is $F_n$ odd? Prove your answer by induction.

**Solution:** Examining the first few terms, one is led to guess that $F_n$ is even when $n$ is a multiple of 3, and odd when $n$ is not a multiple of 3. To prove this by induction, we first observe that the $n = 0$ and $n = 1$ cases are true (because $F_0 = 0$ is even and $F_1 = 1$ is odd). Then in proving the result for $n \geq 2$, we can assume it for $n - 1$ and for $n - 2$. Recall that we have

$$F_n = F_{n-1} + F_{n-2}.$$

There are now three cases, depending on the remainder of $n$ after division by 3.

If $n \equiv 0 \pmod{3}$ (i.e. $n$ is a multiple of 3), then $n - 1$ and $n - 2$ are not multiples of 3, so $F_{n-1}$ and $F_{n-2}$ are odd, so $F_n$ is even as required.

If $n \equiv 1 \pmod{3}$, then $n - 1$ is a multiple of 3 but $n - 2$ is not, so $F_{n-1}$ is even and $F_{n-2}$ is odd, so $F_n$ is odd as required.

If $n \equiv 2 \pmod{3}$ then $n - 1$ is not a multiple of 3 but $n - 2$ is, so $F_{n-1}$ is odd and $F_{n-2}$ is even, so $F_n$ is odd as required.

This completes the inductive step, and the claim follows by induction.
9. Suppose we want to solve a recurrence relation which is almost a kth-order homogeneous linear recurrence relation, but with an extra constant term C:

\[ a_n = r_1a_{n-1} + r_2a_{n-2} + \cdots + r_ka_{n-k} + C, \quad \text{for all } n \geq k. \]

Let \( p(x) = x^k - r_1x^{k-1} - \cdots - r_k \) be the characteristic polynomial of the homogeneous recurrence relation obtained by omitting \( C \).

(a) Show that any solution \( a_n \) also satisfies the \((k + 1)\)th-order linear homogeneous recurrence relation with characteristic polynomial \((x - 1)p(x)\).

**Solution:** Suppose that the sequence \( a_n \) is a solution of our recurrence relation. Then for any \( n \geq k + 1 \), we have

\[
\begin{align*}
  a_n &= r_1a_{n-1} + r_2a_{n-2} + \cdots + r_ka_{n-k} + C \\
  a_{n-1} &= r_1a_{n-2} + r_2a_{n-3} + \cdots + r_ka_{n-k-1} + C.
\end{align*}
\]

Subtracting the second equation from the first gives

\[
a_n - a_{n-1} = r_1(a_{n-1} - a_{n-2}) + r_2(a_{n-2} - a_{n-3}) + \cdots + r_k(a_{n-k} - a_{n-k-1})
\]

for all \( n \geq k + 1 \), which can be rearranged as

\[
a_n = (r_1 + 1)a_{n-1} + (r_2 - r_1)a_{n-2} + \cdots + (r_k - r_{k-1})a_{n-k} + (-r_k)a_{n-k-1}.
\]

This is the homogeneous recurrence relation with characteristic polynomial

\[
x^{k+1} - (r_1 + 1)x^k - (r_2 - r_1)x^{k-1} - \cdots - (r_k - r_{k-1})x + r_k
\]

\[
= (x - 1)(x^k - r_1x^{k-1} - \cdots - r_{k-1}x - r_k) = (x - 1)p(x),
\]

as claimed.

(b) Hence describe the general solution \( a_n \) in terms of the roots of \( p(x) \). (The answer will depend on whether 1 is a root of \( p(x) \) or not.)

**Solution:** Let the different roots of \( p(x) \) be \( \lambda_1, \cdots, \lambda_s \) with multiplicities \( m_1, \cdots, m_s \) (where the multiplicity of a non-repeated root is 1).

First suppose that none of the \( \lambda_i \)'s equals 1; then \((x - 1)p(x)\) has roots \( \lambda_1, \cdots, \lambda_s, 1 \) with multiplicities \( m_1, \cdots, m_s, 1 \). By the general solution of homogeneous recurrence relations given in lectures, (a) implies that

\[
a_n = (C_{11} + C_{12}n + \cdots + C_{1m_1}n^{m_1-1})\lambda_1^n + \cdots + (C_{s1} + C_{s2}n + \cdots + C_{sm_s}n^{m_s-1})\lambda_s^n + D,
\]

for some constants \( C_{ij}, D \). Conversely, any sequence \( a_n \) of this form is a solution of the homogeneous recurrence relation with characteristic polynomial \((x - 1)p(x)\). This is not quite enough to imply that it satisfies our recurrence relation: but the additional requirement is just the \( n = k \) case, namely

\[
a_k = r_1a_{k-1} + \cdots + r_{k-1}a_1 + r_ka_0 + C,
\]

because, as we saw in the previous part, the difference between this equation and the \( n = k + 1 \) case is a case of the homogenous recurrence relation, as is the difference between the \( n = k + 1 \) case and the \( n = k + 2 \) case, and so on.
It is easy to see that the \( n = k \) case reduces to a constraint on the constant \( D \), namely

\[
D = r_1 D + \cdots + r_k D + C,
\]

so the general solution is given by the above formula but with \( D \) specified to equal \( \frac{C}{1 - r_1 - \cdots - r_k} \). (The denominator is \( p(1) \), which we assumed to be nonzero.)

Now suppose that 1 is a root of \( p(x) \); without loss of generality, \( \lambda_s = 1 \). Then \( (x - 1)p(x) \) has roots \( \lambda_1, \ldots, \lambda_s-1, 1 \) with multiplicities \( m_1, \ldots, m_s-1, m_s+1 \). Solving the homogeneous recurrence, we obtain

\[
a_n = (C_{11} + C_{12} n + \cdots + C_{1,m_1} n^{m_1-1}) \lambda_1^n + \cdots \\
+ (C_{s-1,1} + C_{s-1,2} n + \cdots + C_{s-1,m_{s-1}} n^{m_{s-1}-1}) \lambda_{s-1}^n \\
+ (C_{s1} + C_{s2} n + \cdots + C_{s,m_s} n^{m_s-1} + D n^{m_s}),
\]

for some constants \( C_{ij}, D \). As in the previous case, we have one extra constraint on the constant \( D \) in order that the \( n = k \) case of the desired recurrence relation should hold, namely

\[
D k^{m_s} = r_1 D (k-1)^{m_s} + \cdots + r_{k-1} D + C,
\]

so the general solution is given by the above formula but with \( D \) specified to equal \( \frac{C}{k^{m_s} - r_1 (k-1)^{m_s} - \cdots - r_{k-1}} \). (The denominator is nonzero, because it is what you get when you substitute \( x = 1 \) in the polynomial obtained from \( p(x) \) by applying \( m_s \) times the operator \( x \frac{d}{dx} \); each application reduces the multiplicity of the root 1 by 1, so it is no longer a root at the end.)