This assignment comprises a total of 60 marks and is worth 15% of the overall assessment. It should be completed, accompanied by a signed cover sheet, and a hardcopy handed in at the lecture on Wednesday 25 May. Acknowledge any sources or assistance. An electronic copy or scan should also be downloaded using Turnitin from the Blackboard portal.

1. Let $W_1$ and $W_2$ be wffs such that the following sequent can be proved using the 10 rules of deduction in the Propositional Calculus:

$$W_1 \vdash W_2.$$ 

Let $W$ be another wff in the Propositional Calculus. Use Sequent Introduction to prove the following sequents:

(a) $\sim W_2 \vdash \sim W_1$

(b) $W \lor W_1 \vdash W \lor W_2$

Decide which of the following sequents can be proved, providing a proof or counterexample in each case:

(c) $W_2 \Rightarrow W \vdash W_1 \Rightarrow W$

(d) $W_1 \Rightarrow W \vdash W_2 \Rightarrow W$

(9 marks)

2. Use the rules of deduction in the Predicate Calculus (but avoiding derived rules) to find formal proofs for the following sequents:

(a) $(\exists x) F(x) \vdash \sim (\forall x) \sim F(x)$

(b) $\sim (\forall x) \sim F(x) \vdash (\exists x) F(x)$

(c) $(\forall x)(\sim F(x) \Rightarrow G(x)) \vdash ((\exists x) \sim G(x)) \Rightarrow ((\exists y) F(y))$

(d) $(\exists z)(\exists y)(\forall x) K(x, y, z) \vdash (\forall x)(\exists y)(\exists z) K(x, y, z)$

(e) $(\exists x)(G(x) \land (\forall y) (F(y) \Rightarrow H(y, x)))$, 

$$\left(\forall x\right)\left(G(x) \Rightarrow (\forall y)\left(L(y) \Rightarrow \sim H(y, x)\right)\right) \vdash \left(\forall x\right)\left(F(x) \Rightarrow \sim L(x)\right)$$

(20 marks)
3. Find faults in the following arguments, with brief explanations:

(a) First faulty argument:

1 (1) \((\forall x)(F(x) \Rightarrow G(x))\) A
2 (2) \((\exists x) F(x)\) A
3 (3) \(F(a)\) A
1 (4) \(F(a) \Rightarrow G(a)\) \(1 \forall E\)
1, 3 (5) \(G(a)\) \(3, 4 \text{ MP}\)
1, 3 (6) \((\forall x) G(x)\) \(5 \forall I\)
1, 2 (7) \((\forall x) G(x)\) \(2, 3, 6 \exists E\)

(b) Second faulty argument:

1 (1) \((\forall x)(\exists y) H(x, y)\) A
1 (2) \((\exists y) H(a, y)\) \(1 \forall E\)
1 (3) \((\exists y) H(b, y)\) \(1 \forall E\)
4 (4) \(H(a, b)\) A
5 (5) \(H(b, a)\) A
4, 5 (6) \(H(a, b) \land H(b, a)\) \(4, 5 \land I\)
4, 5 (7) \((\exists y)(H(a, y) \land H(y, a))\) \(6 \exists I\)
4, 5 (8) \((\exists x)(\exists y)(H(x, y) \land H(y, x))\) \(7 \exists I\)
1, 4 (9) \((\exists x)(\exists y)(H(x, y) \land H(y, x))\) \(3, 5, 8 \exists E\)
1 (10) \((\exists x)(\exists y)(H(x, y) \land H(y, x))\) \(2, 4, 9 \exists E\)

Now find models to demonstrate that the following sequents are not valid, with brief explanations:

(c) \((\forall x)(F(x) \Rightarrow G(x)), (\exists x) F(x) \vdash (\forall x) G(x)\)

(d) \((\forall x)(\exists y) H(x, y) \vdash (\exists x)(\exists y)(H(x, y) \land H(y, x))\)

(9 marks)

4. Solve the following equations simultaneously over \(\mathbb{Z}_7\) and explain why no solution exists in \(\mathbb{Z}_{11}\):

\[
5x + 2y = 4 \\
3x - y = 3
\]

(4 marks)
5. In this exercise we work with polynomials over \( \mathbb{Z}_3 \). Consider the ring

\[
R = \{0, 1, 2, x, x + 1, x + 2, 2x, 2x + 1, 2x + 2\}
\]

of remainders with addition and multiplication modulo the quadratic

\[
p(x) = x^2 + 2x + 2 = x^2 - x - 1,
\]

where all coefficients come from \( \mathbb{Z}_3 \).

(a) Verify that \( p(x) \) has no linear factors, so is irreducible. (Hence \( R \) is a field.)

(b) Calculate in \( R \) the following powers of \( x \):

\[
x^2, x^3, x^4, x^5, x^6, x^7, x^8.
\]

(c) Explain why \( x \) is primitive in \( R \), but \( x^2 \) is not primitive.

(d) Find both square roots of 2 in \( R \).

(e) Solve over \( R \) the following quadratic equation in \( \alpha \):

\[
\alpha^2 - 2x\alpha + x - 1 = 0.
\]

(9 marks)

6. Suppose that \( a, b, c \in \mathbb{R} \) with \( a \neq 0 \) and \( b^2 - 4ac < 0 \), so that

\[
r(x) = ax^2 + bx + c
\]

is an irreducible quadratic polynomial. Prove that

\[
\mathbb{R}[x]/r(x)\mathbb{R}[x] \cong \mathbb{C}.
\]

[Hint: use the Fundamental Homomorphism Theorem. You may assume without proof that an appropriate evaluation map is a ring homomorphism.]

(9 marks)