1. Let the universe of discourse be the collection of students in the class. Let $F(x)$ denote “$x$ is fierce”, $G(x)$ denote “$x$ is friendly”, $H(x)$ denote “$x$ is smiling” and $E(x, y)$ denote “$x$ and $y$ are the same person”. The statements translate into the following symbolic forms, taking into account the use of the plural in the third and fourth statements:

(a) $(\forall x) \left( F(x) \lor G(x) \right)$

(b) $(\forall x) \left( G(x) \rightarrow H(x) \right)$

(c) $(\exists x)(\exists y) \left[ \left( \sim H(x) \land \sim H(y) \right) \land \sim E(x, y) \right]$

(d) $(\exists x)(\exists y) \left[ \left( F(x) \land F(y) \right) \land \sim E(x, y) \right]$

To see that (d) follows logically from (a), (b) and (c), we apply the rules of Predicate Calculus and the well-known sequent $P \lor Q, \sim Q \vdash P$, which is a separate exercise in the Propositional Calculus, and referenced below as (*):

1. (1) $(\forall x) \left( F(x) \lor G(x) \right)$
2. (2) $(\forall x) \left( G(x) \Rightarrow H(x) \right)$
3. (3) $(\exists x)(\exists y) \left[ \left( \sim H(x) \land \sim H(y) \right) \land \sim E(x, y) \right]$ A
4. (4) $(\exists y) \left[ \left( \sim H(a) \land \sim H(y) \right) \land \sim E(a, y) \right]$ A
5. (5) $(\sim H(a) \land \sim H(b)) \land \sim E(a, b)$ A
6. (6) $F(a) \lor G(a)$ 1 $\forall E$
7. (7) $F(b) \lor G(b)$ 1 $\forall E$
8. (8) $G(a) \Rightarrow H(a)$ 2 $\forall E$
9. (9) $G(b) \Rightarrow H(b)$ 2 $\forall E$
10. (10) $\sim H(a) \land \sim H(b)$ 5 $\land E$
11. (11) $\sim H(a)$ 10 $\land E$
12. (12) $\sim H(b)$ 10 $\land E$
13. (13) $\sim G(a)$ 8, 11 MT
14. (14) $\sim G(b)$ 9, 12 MT
15. (15) $F(a)$ 6, 13 SI(S)(*)
16. (16) $F(b)$ 7, 14 SI(S)(*)
17. (17) $F(a) \land F(b)$ 15, 16 $\land I$
18. (18) $\sim E(a, b)$ 5 $\land E$
19. (19) $(F(a) \land F(b)) \land \sim E(a, b)$ 17, 18 $\land I$
20. (20) $(\exists y) \left[ \left( F(a) \land F(y) \right) \land \sim E(a, y) \right]$ 19 $\exists I$
21. (21) $(\exists x)(\exists y) \left[ \left( F(x) \land F(y) \right) \land \sim E(x, y) \right]$ 20 $\exists I$
22. (22) $(\exists x)(\exists y) \left[ \left( F(x) \land F(y) \right) \land \sim E(x, y) \right]$ 4, 5, 21 $\exists E$
23. (23) $(\exists x)(\exists y) \left[ \left( F(x) \land F(y) \right) \land \sim E(x, y) \right]$ 3, 4, 22 $\exists E$
2. (a) **Claim:** $(\exists x)(\exists y) F(x, y) \not\vdash (\exists x) F(x, x)$

   **Proof:** Consider the universe of discourse $U = \{1, 2\}$ and the relation $F = \{(1, 2)\}$. Then $(1, 2) \in F$, but $(1, 1), (2, 2) \notin F$. Thus $F(1, 2)$ is true, so that $(\exists x)(\exists y) F(x, y)$ is true, whilst $F(1, 1)$ and $F(2, 2)$ are false, so that $(\exists x) F(x, x)$ is false. By Soundness of the Predicate Calculus, the conclusion cannot be deduced from the given hypothesis.

   Line (4) to line (5) of the given “proof” is invalid, because $a$ appears in (2).

(b) **Claim:** $(\forall y)(\exists x) F(x, y) \not\vdash (\exists x)(\forall y) F(x, y)$

   **Proof:** Consider again the universe of discourse $U = \{1, 2\}$ and the relation $F = \{(1, 1), (2, 2)\}$. Then $(1, 1), (2, 2) \in F$, but $(1, 2), (2, 1) \notin F$. Thus $F(1, 1)$ is true, so that $(\exists x) F(x, 1)$ is true, and $F(2, 2)$ is true, so that $(\exists x) F(x, 2)$ is true. Thus $(\forall y)(\exists x) F(x, y)$ is true. But $F(1, 2)$ is false, so that $(\forall y) F(1, y)$ is false, and $F(2, 1)$ is false, so that $(\forall y) F(2, y)$ is false. Hence $(\exists x)(\forall y) F(x, y)$ is false. By Soundness of the Predicate Calculus, the conclusion cannot be deduced from the given hypothesis.

   Line (3) to line (4) of the given “proof” is invalid, because $a$ appears in (3).

3. (a) **Claim:** $(\exists x) F(x, x) \vdash (\exists x)(\exists y) F(x, y)$

   **Proof:**
   
   $\begin{array}{c|c|c}
   1 & (\exists x) F(x, x) & A \\
   2 & F(a, a) & A \\
   \hline
   2 & (\exists y) F(a, y) & 2 \not\vdash I \\
   2 & (\exists x)(\exists y) F(x, y) & 3 \not\vdash I \\
   1 & (\exists y)(\exists x) F(x, y) & 1, 2, 4 \not\vdash E \\
   \end{array}$

   (b) **Claim:** $(\forall x)(\forall y)(\forall z) F(x, y, z) \vdash (\forall z)(\forall y)(\forall x) F(x, z, y)$

   **Proof:**
   
   $\begin{array}{c|c|c}
   1 & (\forall x)(\forall y)(\forall z) F(x, y, z) & A \\
   1 & (\forall y)(\forall z) F(a, y, z) & 1 \forall E \\
   1 & (\forall z) F(a, b, z) & 2 \forall E \\
   1 & F(a, b, c) & 3 \forall E \\
   1 & (\forall x) F(x, b, c) & 4 \forall I \\
   1 & (\forall y)(\forall x) F(x, b, y) & 5 \forall I \\
   1 & (\forall z)(\forall y)(\forall x) F(x, z, y) & 6 \forall I \\
   \end{array}$

   *(c) **Claim:** $(\forall x)(\exists y)(\exists z) F(x, y, z) \vdash (\forall x)(\forall y)(\exists z) F(x, z, y)$

   **Proof:**
   
   $\begin{array}{c|c|c}
   1 & (\forall x)(\exists y)(\exists z) F(x, y, z) & A \\
   1 & (\exists y)(\forall z) F(a, y, z) & 1 \forall E \\
   3 & (\forall z) F(a, b, z) & A \\
   3 & F(a, b, c) & 3 \forall E \\
   3 & (\exists z) F(a, z, c) & 4 \exists I \\
   1 & (\exists z) F(a, z, c) & 2, 3, 5 \not\exists E \\
   1 & (\forall y)(\exists z) F(a, z, y) & 6 \forall I \\
   1 & (\forall x)(\forall y)(\exists z) F(x, z, y) & 7 \forall I \\
   \end{array}$
4. (a) Let $U = \mathbb{Z}^+$, and for all $x, y \in \mathbb{Z}^+$, interpret $R(x, y)$ to mean $x < y$. Certainly, for each $x \in \mathbb{Z}^+$, $x \not< x$, so that $W_1$ holds in $U$. Furthermore, $<$ is transitive, that is, for all $x, y, z \in \mathbb{Z}^+$,

$$x < y < z \Rightarrow x < z,$$

so that $W_2$ holds in $U$. Finally, for all $x, y \in \mathbb{Z}^+$, we can take $z = \max\{x, y\} + 1$, and it is certainly true that

$$x < z \text{ and } y < z,$$

so that $W_3$ holds in $U$. This proves that $U$ serves as a model for $W_1, W_2, W_3$.

(b) Let $U$ be any model for $W_1, W_2, W_3$. Certainly we can find $x_1 \in U$, since $U$ is nonempty. Because $W_3$ holds in $U$, taking $x = y = x_1$, we know there exists $x_2 \in U$ such that

$$R(x, x_2) \text{ and } R(y, x_2),$$

so that, certainly, $R(x_1, x_2)$. Note that $x_1$ is different from $x_2$, because $W_1$ holds. This start an inductive process. Suppose, as an inductive hypothesis, that $n \geq 2$ is a positive integer and that we have found elements $x_1, \ldots, x_n \in U$, all distinct, such that

$$R(x_1, x_2), \ R(x_2, x_3), \ldots, \ R(x_{n-1}, x_n).$$

Because $W_3$ holds in $U$, taking $x = y = x_n$, we know there exists $x_{n+1} \in U$ such that

$$R(x, x_{n+1}) \text{ and } R(y, x_{n+1}),$$

so that, certainly, $R(x_n, x_{n+1})$. Hence we have

$$R(x_1, x_2), \ R(x_2, x_3), \ldots, \ R(x_{n-1}, x_n), \ R(x_n, x_{n+1}).$$

Suppose that $x_{n+1} = x_m$ for some $m \leq n$. But we have

$$R(x_m, x_{m+1}), \ R(x_{m+1}, x_{m+2}), \ldots, \ R(x_{n}, x_{n+1}).$$

By applying $W_2$ repeatedly, we get $R(x_m, x_{n+1})$, so that

$$R(x_m, x_m),$$

which contradicts $W_1$. This proves that $x_1, \ldots, x_{n+1}$ are all distinct. By induction, we have a list of distinct elements

$$x_1, \ x_2, \ \ldots, \ x_n, \ \ldots$$

of $U$ (in fact an injective map from $\mathbb{Z}^+$ into $U$), which proves $U$ is infinite. Thus no finite model of $W_1, W_2, W_3$ exists.