Tutorial Exercises: Calculus of Variations

1. The Catenoid

Consider the integrand $F(x, y, y') = y\sqrt{1 + (y')^2}$ in Eq. (1.5) when $y$ is a function of $x$.

(a) Determine the Lagrange equation.

(b) There is a first integral; write it down and rearrange to make $y'$ the subject.

(c) Solve the first-order differential equation by separating variables and integrating.

1. Solution: The Catenoid

(a) The Lagrange equation is

$$\frac{d}{dx} \left( \frac{\partial F}{\partial y'} \right) - \frac{\partial F}{\partial y} = \frac{d}{dx} \left( \frac{yy'}{\sqrt{1 + (y')^2}} \right) - \sqrt{1 + (y')^2} = y' + \frac{yy''}{\sqrt{1 + (y')^2}} - \sqrt{1 + (y')^2} = 0$$

This can be simplified to

$$\frac{yy'' - 1 - y'y'}{\sqrt{1 + (y')^2}} = 0.$$

Since the denominator cannot be zero this gives the 2nd order ODE

$$yy'' - 1 - y'^2 = 0.$$  

(b) Since $F$ does not depend explicitly on $x$ there is a first integral

$$y' \frac{\partial F}{\partial y'} - F = y' \frac{yy'}{\sqrt{1 + (y')^2}} - y\sqrt{1 + (y')^2} = C$$

Rearranging gives the 1st order ODE

$$y' = \pm \sqrt{\frac{y^2 - C^2}{C^2}}.$$  

(c) Separating variables and integrating gives

$$\int \frac{dy}{\sqrt{y^2 - C^2}} = \pm \int \frac{dx}{C}$$

A good substitution to try is $y = C \cosh(\alpha)$ then $dy = C \sinh(\alpha) d\alpha$, thus

$$\int \frac{C \sinh(\alpha) d\alpha}{C \sinh(\alpha)} = \pm \frac{x + D}{C}.$$  

The l.h.s gives $\alpha = \cosh^{-1}(y/C)$ and therefore

$$y = C \cosh(\frac{x + D}{C}).$$

2. Dido’s Problem

Consider the integrand $F(s, y, y') = y\sqrt{1 - (y')^2}$ in Eq. (1.10) when $y$ is a function of $s$.

(a) Use a first integral to find an expression for $y'$.

(b) Solve the first-order differential equation by separation.

(c) Apply the boundary conditions $y = 0$ at both $s = 0$ and $s = L$ to eliminate some constants.

(d) Find an equation for $dx/ds$ and then find $x(s)$.

(e) What geometric curve does this parametric solution represent?
2. Solution: Dido’s Problem

(a) Since $F$ does not depend on $s$ a first integral is

$$y_y \frac{\partial F}{\partial y'} - F = y' y \frac{-2y'}{2\sqrt{1 - (y')^2}} - y \sqrt{1 - (y')^2} = C$$

Rearranging gives

$$y = -C \sqrt{1 - (y')^2}$$

and solving for $y'$ gives the explicit 1st order ODE

$$y' = \pm \frac{\sqrt{C^2 - y^2}}{C}.$$

(b) Separating variables gives

$$\int \frac{C \, dy}{\sqrt{C^2 - y^2}} = \pm \int ds.$$ 

The integral on the left is an inverse trig function so that

$$C \sin^{-1} \left( \frac{y}{C} \right) = \pm (s + D)$$

Thus

$$y = \pm C \sin \frac{s + D}{C}.$$

(c) The boundary conditions give

$$0 = \pm C \sin \frac{D}{C}$$

and

$$0 = \pm C \sin \frac{L + D}{C}.$$ 

All formal solutions are given by $D/C = n\pi$ and $L/C + n\pi = m\pi$ so that $C = L/(k\pi)$ and $D = Ln/k$ for integers $n,m,k$. But note that $D$ is merely shifting $s$, so we can choose $D = 0$. The simplest solution is $k = 1$ so that $C = \frac{L}{\pi}$. Choosing $|k| > 1$ we do not obtain a positive $y$ for all $s \in [0,L]$. Thus,

$$y = \frac{L}{\pi} \sin \frac{s\pi}{L}.$$

(d) Now

$$\frac{dy}{ds} = \cos \frac{s\pi}{L}$$

Thus from

$$\left( \frac{dx}{ds} \right)^2 + \left( \frac{dy}{ds} \right)^2 = 1$$

we can determine that

$$\frac{dx}{ds} = \pm \sin \frac{s\pi}{L}$$

Thus, by integrating

$$x(s) = \pm \frac{L}{\pi} \cos \frac{s\pi}{L} + C_2$$

The boundary conditions (and choosing $x$ to be positive) lead to

$$x(s) = -\frac{L}{\pi} \cos \frac{s\pi}{L} + \frac{L}{\pi}.$$ 

There is no endpoint condition on $x$ but instead the $x$-coordinate of the endpoint of the optimal curve is determined by $x(L) = 2L/\pi$. 

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(e) This is a semicircle of radius \( \frac{L}{\pi} \) with centre at \((\frac{L}{\pi}, 0)\). The area of the city is \( \frac{L^2}{2(2\pi)} \). Notice that the first integral can now be retrospectively interpreted as the radius of the city. First integrals often have interesting interpretations.

3. The Catenary

(a) Determine \( F(x, y, y') \) for the integrand in Eq. (1.12) when \( y \) is a function of \( x \).
(b) Solve the corresponding Euler-Lagrange equations or the first integral if there is one.
(c) Apply the boundary conditions \( y = h \) at both \( x = -a \) and \( x = a \) to determine the constants of integration.

3. Solution: The Catenary

(a) The integral is
\[
\int_A^B (\rho y + \lambda) ds = \rho \int_A^B (y + \frac{\lambda}{\rho}) ds
\]
but since \( \lambda \) is arbitrary there is not much point in writing \( \frac{\lambda}{\rho} \) instead of writing just \( \lambda \) so one can do the calculation using \((y + \lambda) ds\).
When \( y \) is a function of \( x \) this gives
\[
F(x, y, y') = (y + \lambda)\sqrt{1 + y'^2}.
\]
(b) Since \( F \) does not depend on \( x \) the appropriate first integral is
\[
y' \frac{\partial F}{\partial y'} - F = y'(y + \lambda)y' - (y + \lambda)\sqrt{1 + y'^2} = C
\]
or
\[
-(y + \lambda) = C.
\]
Thus,
\[
y'^2 = \frac{(y + \lambda)^2 - C^2}{C^2}.
\]
Solving by separation gives
\[
\int \frac{dy}{\sqrt{(y + \lambda)^2 - C^2}} = \pm \int \frac{dx}{C}
\]
and a good substitution is \( y + \lambda = C \cosh \alpha \) or \( dy = C \sinh \alpha \, d\alpha \) and so
\[
\alpha = \pm \left( \frac{x}{C} + D \right).
\]
The solution is \( y = C \cosh \left[(x/C) + D\right] - \lambda \).
(c) The boundary conditions \( x = -a \) and \( x = a \) can be used to eliminated \( D \) and \( \lambda \). Even though \( \lambda \) was introduced to control \( L \) at this stage we may use it to satisfy the other boundary conditions. Hence we get
\[
y = C \cosh \left[ \frac{x}{C} \right] - C \cosh \left[ \frac{a}{C} \right] + h.
\]
The final remaining unknown \( C \) is determined by the length of the catenary
\[
L = \int ds = \int_{x=-a}^{a} \sqrt{1 + \sinh^2 \left( \frac{x}{C} \right)} \, dx = \int_{x=-a}^{a} \cosh \left( \frac{x}{C} \right) \, dx = \left[ C \sinh \left( \frac{x}{C} \right) \right]_{x=-a}^{a} = 2C \sinh \left( \frac{a}{C} \right).
\]
Unfortunately, the final result cannot be put in a form more explicit than that.
4. Calculating $ds$ in a different coordinate system

Cylindrical polar coordinates are defined by

\[
\begin{align*}
x &= \rho \cos \phi \\
y &= \rho \sin \phi \\
z &= z
\end{align*}
\]

(a) Confirm that $dx = d\rho \cos \phi - \rho \sin \phi d\phi$.

(b) Calculate a similar expression for $dy$.

(c) Starting from $ds^2 = dx^2 + dy^2 + dz^2$ show that $ds^2 = d\rho^2 + \rho^2 d\phi^2 + dz^2$.

(d) Having warmed up with that calculation, repeat with spherical polar coordinates which are defined by

\[
\begin{align*}
x &= r \sin \theta \cos \phi \\
y &= r \sin \theta \sin \phi \\
z &= r \cos \theta
\end{align*}
\]

and show that $ds^2 = dr^2 + r^2 d\theta^2 + r^2 \sin^2 \theta d\phi^2$.

**Hint:** The spherical result is easier to get starting from the cylindrical result and using $\rho = r \sin \theta$.

4. Solution: Calculating $ds$ in a different coordinate system

(a) This is a simple application of the product rule $dx = d\rho \cos \phi - \rho \sin \phi d\phi$.

(b) $dy = d\rho \sin \phi + \rho \cos \phi d\phi$.

(c) Now

\[
dx^2 + dy^2 = (d\rho \cos \phi - \rho \sin \phi d\phi)^2 + (d\rho \sin \phi + \rho \cos \phi d\phi)^2
\]

The cross terms cancel so

\[
dx^2 + dy^2 = d\rho^2 (\cos^2 \phi + \sin^2 \phi) + \rho^2 d\phi^2 (\sin^2 \phi + \cos^2 \phi) = d\rho^2 + \rho^2 d\phi^2
\]

Adding $dz^2$ gives the desired result.

(d) Start with $ds^2 = d\rho^2 + \rho^2 d\phi^2 + dz^2$. Now $\rho^2 = r^2 \sin^2 \theta$ and the product rule applied to $d\rho$ gives

\[
d\rho = dr \sin \theta + r \cos \theta d\theta
\]

and also

\[
dz = dr \cos \theta - r \sin \theta d\theta
\]

Thus substituting into $ds^2$ from above gives

\[
ds^2 = (dr \sin \theta + r \cos \theta d\theta)^2 + r^2 \sin^2 \theta d\phi^2 + (dr \cos \theta - r \sin \theta d\theta)^2
\]

Once again the cross-terms cancel leaving

\[
ds^2 = dr^2 (\sin^2 \theta + \cos^2 \theta) + r^2 (\cos^2 \theta + \sin^2 \theta) d\theta^2 + r^2 \sin^2 \theta d\phi^2
\]

which simplifies to the desired result $ds^2 = dr^2 + r^2 d\theta^2 + r^2 \sin^2 \theta d\phi^2$. 


5. **Geodesics on the Sphere**

The equation of a sphere in spherical polar coordinates is particularly simple: it is \( r = a \), where \( a \) is a constant.

(a) Starting with \( ds \) in spherical polar coordinates, write down the simplified form of \( ds \) when \( r = a \) is a constant.

(b) Use this expression for \( ds \) to write down an integral that represents the distance between two points connected by a path that lies on the surface of a sphere. Write the integral in the form where \( \phi \) is a function of \( \theta \).

(c) Write down a first integral for this integrand.

(d) Show that
\[
\phi - \phi_0 = \sin^{-1}[\alpha \cot \theta]
\]
satisfies the first integral, where \( \phi_0 \) and \( \alpha \) are two independent constants.

(e) The equation of a plane through the origin is \( Ax + By + Cz = 0 \). Rewrite this equation in spherical polar coordinates. Rearrange the equation to make it look like the solution above and find \( \alpha \) and \( \phi_0 \) in terms of \( A, B \) and \( C \).

(f) Thus give a simple geometric description and method of finding geodesics on a sphere.

5. **Solution: Geodesics on the Sphere**

(a) If \( r = a \) is a constant then
\[
ds^2 = a^2 d\theta^2 + a^2 \sin^2 \theta \, d\phi^2.
\]

(b) The integral is
\[
I = \int ds = a \int_{\theta_0}^{\theta_B} \sqrt{1 + \sin^2 \theta \phi'^2} \, d\theta.
\]
Thus
\[
F(\theta, \phi, \phi') = \sqrt{1 + \sin^2 \theta \phi'^2}.
\]
(c) Since \( \partial F / \partial \phi = 0 \) a first integral is
\[
\frac{\sin^2 \theta \phi'}{\sqrt{1 + \sin^2 \theta \phi'^2}} = C
\]
or
\[
\phi' = \pm \left( \frac{C}{\sin \theta \sqrt{\sin^2 \theta - C^2}} \right).
\]
(d) Direct differentiation and some algebra, yields the result.

(e) In spherical coordinates this becomes
\[
Ar \sin \theta \cos \phi + Br \sin \theta \sin \phi + Cr \cos \theta = 0
\]
The \( r \) cancels, the \( \cos \theta \) can be moved to the other side and both sides divided by \( \sin \theta \) to give
\[
A \cos \phi + B \sin \phi = -C \cot \theta
\]
Trig identities can be used to rewrite the left hand side as
\[
\sqrt{A^2 + B^2} \sin(\phi - \phi_0) = -C \cot \theta
\]
where \( \phi_0 = -\tan^{-1}(A/B) \) and \( \alpha = -C/\sqrt{A^2 + B^2} \).

(f) In other words, the curve with the shortest distance lies simultaneously on the surface of a sphere AND on a plane through the origin. The intersection of such a plane and a sphere is called a *great circle*. 
6. Geodesics in three dimensions

(a) Determine $ds^2$ for Cartesian coordinates $x, y, z$ in three dimensions.
(b) Write the integrand for the geodesic problem when both $y$ and $z$ are functions of $x$.
(c) Write down all the first integrals. (There are three.)
(d) Pick two of the first integrals (presumably the nicest two) and solve the system of differential equations.
(e) Interpret the solution geometrically.

6. Solution: Geodesics in three dimensions

(a) $ds^2 = dx^2 + dy^2 + dz^2$

(b) $F(x, y, z, y', z') = \sqrt{1 + y'^2 + z'^2}$.

(c) The integrand is independent of $y$ thus

$$\frac{y'}{\sqrt{1 + y'^2 + z'^2}} = C_1.$$  

The integrand is independent of $z$ thus

$$\frac{z'}{\sqrt{1 + y'^2 + z'^2}} = C_2.$$  

The integrand is independent of $t$ thus

$$\frac{y^2}{\sqrt{1 + y'^2 + z'^2}} + \frac{z'^2}{\sqrt{1 + y'^2 + z'^2}} - \sqrt{1 + y'^2 + z'^2} = C_3$$

which is the same as

$$-\frac{1}{\sqrt{1 + y'^2 + z'^2}} = C_3$$

(d) These can be rearranged to give $y' = A$ a constant, and $z' = B$ a constant. Thus $y = Ax + C$ and $z = Bx + D$.

(e) This is a straight line in 3 dimensions.

7. The Brachistochrone in Parametric Form

(a) Determine the integrand $F(t, x, y, \dot{x}, \dot{y})$ for Eq. (1.6) when both $x$ and $y$ are functions of some parameter $t$.
(b) Write down all the first integrals. Are any surprising?
(c) Show that the parametric form of the cycloid given in the text solves these equations.
(d) Find a different pair of functions $x(t)$ and $y(t)$ that also solve the same equations.
(e) Why should this be regarded as the same solution?
7. Solution: The Brachistochrone in Parametric Form

(a) 
\[ F(t, x, y, \dot{x}, \dot{y}) = \frac{\sqrt{\dot{x}^2 + \dot{y}^2}}{\sqrt{h - y}} \]

(b) The integrand is independent of \( x \) so
\[ \frac{\partial F}{\partial \dot{x}} = \frac{\dot{x}}{\sqrt{h - y} \sqrt{\dot{x}^2 + \dot{y}^2}} = C \]

The integrand is independent of \( t \) so
\[ \dot{x} \frac{\partial F}{\partial \dot{x}} + \dot{y} \frac{\partial F}{\partial \dot{y}} - F = \frac{\dot{x}^2}{\sqrt{h - y} \sqrt{\dot{x}^2 + \dot{y}^2}} + \frac{\dot{y}^2}{\sqrt{h - y} \sqrt{\dot{x}^2 + \dot{y}^2}} - \frac{\sqrt{\dot{x}^2 + \dot{y}^2}}{\sqrt{h - y}} = 0 = \text{a constant} \]

The information that 0 is a constant is true but not very useful!

(c) Let the parameter \( t \) be the same as \( \theta \) in the text. Then \( \dot{x} = \alpha^2 - \alpha^2 \cos(2t) \) and \( \dot{y} = -\alpha^2 \sin(2t) \) and substituting this into the first integral for \( x \) gives
\[ \frac{\alpha^2 - \alpha^2 \cos(2t)}{\alpha \sin(t) \sqrt{2\alpha^4 - 2\alpha^4 \cos(2t)}} = C \]

which simplifies to
\[ \frac{1 - \cos(2t)}{\alpha \sin(t) \sqrt{2 - 2 \cos(2t)}} = C \]
or
\[ \frac{1}{\alpha \sqrt{2}} = C \]

(d) You can let \( \theta \) equal any function of \( t \) and the equations will still be satisfied. This is because there are an equal number of dots in the top and bottom of the LHS of the first integral.

(e) The parametric form of a curve is not unique, but they all represent the same curve.