I. Examples of optimisation problems

A. Optimisation in Physical World

Every physical system evolves so as to minimize its energy.

Light chooses its path so as to reach the target in minimal time.

B. Animals optimize

Pursuit problem: a dog is chasing a rabbit. For simplicity, assume that rabbit runs with a constant speed along a straight line.

Dog chooses the optimal curve to catch the rabbit, it minimizes pursuit time needed to catch the rabbit whatever it does.

C. Engineers optimize

a) Start of a rocket.
How to choose the engine force as a function of time in such a way the elevation $H$ is achieved with minimal fuel consumption or in military applications, in minimal time.

b) Traveling salesman problem: $n$ cities

Find the route visiting all cities (shortest)

D Economists optimize

Portfolio optimisation

I have $1000\$ and want to invest. Take $0 \leq x \leq 1$. I can invest $1000x \$ into savings account with fixed interest rate $r$. At the end of the period I will receive $1000x(1+r)\$. I can buy BHP shares for $1000(1-x)\$. But shares evolve randomly

$\frac{6}{2}S$

$S \xrightarrow{\frac{4}{2}} \frac{1}{2}S$
How to choose $\alpha$ to maximize value of my capital at the end of the period?

How to choose $\alpha$ in each period if there are many periods.
Revision: optimisation for functions of one variable

What can happen?

- $-\infty \leq a < b \leq \infty$

\[ f(a) = \min_{a \leq x \leq b} f(x) \]
\[ f(b) = \max_{a \leq x \leq b} f(x) \]

unique global maximum in the interior

unique global minimum at \( a \)

many local minima and maxima

global maxima at \( a \), global minimum at \( c \)
D ⊆ \mathbb{R} \text{ subset of real line}

\( f : D \rightarrow \mathbb{R} \)

A point \( x^* \) is a **global minimum** of \( f \) if
\[
 f(x^*) \leq f(x) \quad \text{all } x \in D
\]

A point \( \alpha \) is a **local minimum** for \( f \) if
\( \alpha \) is in the interior of \( D \) and for a small interval \( (\alpha, \beta) \subseteq D \) around \( \alpha \) we have
\[
 f(\alpha) \leq f(x) \quad x \in (\alpha, \beta)
\]

**Similar definitions for maxima**

**Extremum** (or extremal point) = minimum or maximum

**Critical point** = any solution of the equation
\[
 f'(x) = 0
\]

\( \alpha \) in the interior of \( D \)
If \( f^{(1)}(x^*) = \cdots = f^{(m-1)}(x^*) = 0 \)

\[ f^{(m)}(x^*) < 0 \implies \text{maximum} \]

If \( f^{(1)}(x^*) = \cdots = f^{(m-1)}(x^*) = 0 \) and

\[ f^{(m)}(x^*) > 0 \implies \text{minimum} \]
Example:

\[
\begin{align*}
\mathcal{A}(x) &= x^4 \\
\mathcal{B}(x) &= x^3 \\
\mathcal{C}(x) &= 12x^2 \\
\mathcal{D}(x) &= 24x \\
\mathcal{E}(x) &= 24 \\
\mathcal{F}(x) &= 0 \\
\mathcal{G}(x) &= 0 \\
\mathcal{H}(x) &= 0
\end{align*}
\]

\[
\begin{align*}
\mathcal{J}(x) &= e^{-\frac{1}{x^2}} \\
\mathcal{K}(x) &= 0 \\
\mathcal{L}(x) &= 0
\end{align*}
\]

Why the criterion works?

Idea of proof: Taylor polynomial:

\[
\begin{align*}
\mathcal{M}(x) &= \mathcal{M}(x^*) + \frac{\mathcal{M}''(x^*)(x-x^*)^2}{2!} + \frac{\mathcal{M}^{(2m-1)}(x^*)}{(2m-1)!}(x-x^*)^{2m} \\
&\quad + \frac{\mathcal{M}^{(2m)}(x^*)}{(2m)!}(x-x^*)^{2m}
\end{align*}
\]
If \( f^{(1)}(x^*) = \cdots = f^{(m-1)}(x^*) = 0 \)

then

\[
f(x) \approx \frac{f^{(m)}(x^*)}{(2m-1)!} (x-x^*)^{2m} + f(x^*)
\]

If \( f^{(2m)}(x^*) > 0 \)
Functions of many variables

\[ D \subseteq \mathbb{R}^n \quad x \in D \iff x = (x_1, \ldots, x_n) \]

\[ f : D \to \mathbb{R} \]

critical point \( x^* = (x_1^*, \ldots, x_n^*) \)

\[ \frac{\partial f}{\partial x_i}(x^*) = \ldots = \frac{\partial^2 f}{\partial x_i^2}(x^*) = 0 \]

\[ D(x^*) = \begin{bmatrix} \frac{\partial^2 f}{\partial x_i^2} & \frac{\partial^2 f}{\partial x_i \partial x_j} \\ \frac{\partial^2 f}{\partial x_j \partial x_i} & \frac{\partial^2 f}{\partial x_j^2} \end{bmatrix} \]

Note that

\[ \frac{\partial^2 f}{\partial x_i \partial x_j} = \frac{\partial^2 f}{\partial x_j \partial x_i} \]

If all the second derivatives are continuous

\[ n = 2 \]

\[ \frac{\partial^2 f}{\partial x_i^2}(x^*) > 0 \quad \text{Det} \ D(x^*) > 0 \quad \text{minimum} \]

\[ \frac{\partial^2 f}{\partial x_i^2}(x^*) < 0 \quad \text{Det} \ D(x^*) < 0 \quad \text{maximum} \]

\[ \frac{\partial^2 f}{\partial x_i^2}(x^*) < 0 \quad \text{Det} \ D(x^*) < 0 \quad \text{saddle point} \]