1. Suppose $T_n$ for $n \geq 0$ is defined recursively by
   \[ T_0 = 5 \quad \text{and} \quad T_n = 2T_{n-1} + 1 \quad \text{for } n \geq 1 \, . \]
   (a) Find $T_1$, $T_2$, $T_3$, $T_4$ and $T_5$.
   (b) Verify by induction that $T_n = 6(2^n) - 1$
       for all $n \geq 0$.
   *(c) More generally, let $k$, $c$ and $d$ be constants where $c \neq 0, 1$. Verify that if
       $T_0 = k \, , \quad T_n = cT_{n-1} + d$ for $n \geq 1$
       then
       \[ T_n = c^n \left( k - \frac{d}{1-c} \right) + \frac{d}{1-c} \, . \]
       (In the Tower of Hanoi Problem $k = 0$, $c = 2$ and $d = 1$.) What happens if $c = 1$?

2. (a) Verify by induction or otherwise the following formula for sums of even integers:
   \[ 2 + 4 + 6 + \ldots + 2n = n^2 + n \, . \]
   *(b) More generally verify the following formula for the sum of a finite arithmetic sequence:
       \[ a + (a+d) + (a+2d) + \ldots + (a+(n-1)d) = \frac{n}{2}[2a + (n-1)d] \, . \]

3. (a) Verify by induction or otherwise the following formula for sums of powers of 2:
   \[ 1 + 2 + 2^2 + \ldots + 2^n = 2^{n+1} - 1 \, . \]
   *(b) More generally verify the following formula for the sum of a finite geometric sequence when $r \neq 1$:
       \[ a + ar + ar^2 + \ldots + ar^n = \frac{a(1-r^{n+1})}{1-r} \]
       and deduce the following well-known formula for an infinite geometric series when $|r| < 1$:
       \[ \frac{1}{1-r} = 1 + r + r^2 + r^3 + \ldots \]
4. Verify by induction that $2^n < n!$ for all $n \geq 4$.

*5. Prove that, for all $n \geq 1$,

\[
\frac{1}{1 \times 2} + \frac{1}{2 \times 3} + \cdots + \frac{1}{n(n+1)} = \frac{n}{n+1}.
\]

6. Verify that, for all integers $n \geq 1$,

(a) $n^3 + 5n$ is a multiple of 3.

*(b) $5^n - 4n - 1$ is divisible by 16.

7. Verify that

(a) $1^2 + 2^2 + \ldots + n^2 = \frac{n(n+1)(2n+1)}{6}$.

*(b) $(1 + 2 + \ldots + n)^2 = 1^3 + 2^3 + \ldots + n^3$.

**8. A store sells pencils in packs of 5 or 12. You need $n$ pencils. Prove that the store can fill your exact order provided $n \geq 44$. What if $n = 43$?

**9. Find the error in the following argument that claims to prove that all the people in this room have the same height:

We argue by induction on the number $n$ of people in this room. If $n = 1$ then the claim is obviously true, which begins the induction. Suppose $k \geq 1$ and the claim holds for rooms with $k$ people. Suppose $P_1, P_2, \ldots, P_{k+1}$ are the $k+1$ people in this room. By the inductive hypothesis $P_1, P_2, \ldots, P_k$ all have the same height, and also $P_2, \ldots, P_k, P_{k+1}$ all have the same height (since these two groups consisting of $k$ people could each be separated out into a single room). But $P_2$ is in common, so all of $P_1, P_2, \ldots, P_k, P_{k+1}$ have the same height. This establishes the inductive step, and so the claim holds for all $n$ by induction.

**10. Prove that a $2^n \times 2^n$ grid with exactly one subsquare missing can be tiled by $2 \times 2$ squares with one subsquare missing (L-shapes).