1. The products in \( \mathbb{Z}_{11} \) are 6, 1, 7, 2, 8, 3, 9, 4, 10, 5.

*2. We adapt the method in lectures. We have \( 35 = 5 \cdot 7 \) and the following integer linear combinations involving 5 and \( 5 - 1 = 4 \) and \( 7 - 1 = 6 \) respectively:
\[
1 = 1(5) + (-1)(4) = 5(5) + (-4)(6)
\]
Observe that \( 8^1 = 8 \equiv 3 \pmod{5} \) and \( 8^5 = 1^5 = 1 \pmod{7} \) so, to find \( M \), we solve
\[
M = 3 \pmod{5} \quad \text{and} \quad M = 1 \pmod{7}
\]
But \( 3(5) - 2(7) = 1 \), so take \( M' = 1(3(5)) - 3(2(7)) = 15 - 42 = -27 \), so
\[
M = M' = -27 = -27 + 35 = 8 \pmod{35}.
\]
Now observe that \( 4^1 = 4 \pmod{5} \) and \( 4^5 = 64(16) = 1(2) = 2 \pmod{7} \) so, to find \( N \), we solve
\[
N = 4 \pmod{5} \quad \text{and} \quad N = 2 \pmod{7}
\]
But, again, \( 3(5) - 2(7) = 1 \), so take \( N' = 2(3(5)) - 4(2(7)) = 30 - 56 = -26 \), so
\[
N = N' = -26 = -26 + 35 = 9 \pmod{35}.
\]
Hence the \( M \)th and \( N \)th letters combine to say HI!

3. Clearly
\[
A \cup B = \{1, 2, 3, 4, 5, 6\}
\]
\[
A \cup C = \{1, 2, 3, 4, 6, 7\}
\]
\[
B \cap C = \{3, 6\}
\]
\[
A \cup (B \cap C) = \{1, 2, 3, 4, 6\} = (A \cup B) \cap (A \cup C)
\]
\[
C \setminus A = \{6, 7\}
\]
\[
C \setminus B = \{4, 7\}
\]
\[
(C \setminus A) \cup (C \setminus B) = \{4, 6, 7\} = C \setminus (A \cap B)
\]
4. (a) We have

$$|A \cup B| = |A| + |B| - |A \cap B| = 140 + 92 - 36 = 196.$$  

(b) In this case we have

$$|A \cap B| = |A| + |B| - |A \cup B| = 140 + 92 - 150 = 82.$$  

(c) Suppose there exists a set $C$ such that

$$|C| = 58, \quad |A \cap B| = 32, \quad |A \cap B \cap C| = 10, \quad |A \cup B \cup C| = 250.$$  

Then

$$250 = |A \cup B \cup C| = |A| + |B| + |C| - |A \cap B| - |A \cap C| - |B \cap C| + |A \cap B \cap C|$$
$$= 140 + 92 + 58 - 32 - |A \cap C| - |B \cap C| + 10$$
$$= 268 - |A \cap C| - |B \cap C|.$$  

But both $A \cap C$ and $B \cap C$ contain $A \cap B \cap C$, so each have size at least 10. Thus

$$20 \leq |A \cap C| + |B \cap C| \leq 268 - 250 = 18,$$

which is impossible. Hence no such set $C$ exists.

*5. (a) Let $A$, $B$, $C$ be the sets of positive integers $\leq 500$ which are multiples of 5, 7, 11 respectively. We want $|A \cup B \cup C|$. But $|A| = 100$, $|B| = 71$, $|C| = 45$, $|A \cap B| = 14$, $|A \cap C| = 9$, $|B \cap C| = 6$, $|A \cap B \cap C| = 1$. By inclusion-exclusion

$$|A \cup B \cup C| = 100 + 71 + 45 - 14 - 9 - 6 + 1 = 188.$$  

(b) In addition let $D$ be the set of integers $\leq 500$ which are multiples of 3, so we want

$$|(A \cap B) \setminus D| = |A \cap B| - |A \cap B \cap D| = 14 - 4 = 10.$$  

(c) Now let $E$ be the set of integers $\leq 500$ which are multiples of 6, so we want

$$|E \setminus (A \cup B)| = |E| - |A \cap E| - |B \cap E| + |A \cap B \cap E| = 83 - 16 - 11 + 2 = 58.$$  

**6. Using the method of 5(a) we get that the number of integers $\leq 100$ which are divisible by at least one of 2, 3, 5 or 7 must be

$$50 + 33 + 20 + 14 - 16 - 10 - 7 - 6 - 4 - 2 + 3 + 2 + 1 + 0 - 0 = 78.$$
But this includes all composite numbers and 2, 3, 5, 7. Hence there are 74 composite numbers between 2 and 100. Thus there are $99 - 74 = 25$ prime numbers less than 100.

7. There are 26 letters in the alphabet. The function that takes a student to the first letter of his or her surname cannot be one-one if there are 40 students, in which case at least two students have surnames beginning with the same letter.

If there are 80 students in the class then at least 4 students must have surnames beginning with the same letter. To see this, suppose otherwise. Then each of the 26 letters is the first letter of at most 3 surnames, so that there are at most $26 \times 3 = 78 < 80$ students, a contradiction.

*8. Consider the $n + 1$ numbers

$$1, \ 11, \ 111, \ \cdots, \ 111\ldots1,$$

where the last number is a string of $n + 1$ digits. If any of these is a multiple of $n$ then we are done. Suppose none of these is a multiple of $n$. By the pigeonhole principle at least two must be equal modulo $n$. Subtracting the smaller from the larger gives an integer of the form $1\ldots10\ldots0$, and again we are done, since this difference is a multiple of $n$.

*9. Suppose all $k$ boxes contain fewer than $\lceil n/k \rceil$ objects. Then the total number of objects is

$$n \leq k(\lceil n/k \rceil - 1) \leq k\lceil n/k \rceil - k < k(n/k + 1) - k = n,$$

so that $n < n$, which is impossible. Hence at least one box contains $\lceil n/k \rceil$ objects.

**10. (a) The encoded message is

0261|2909|0666|2401|1598|3039

(b) Factorizing 3,763 as a product of primes $p$ and $q$ is a key step to (d) so the solution will be deferred until the chocolate orange has been awarded.

(c) This is straightforward and is a good check on whether you have the method right in order to tackle part (d).

(d) Whoever wins the chocolate orange is more than qualified to help you with the solution!!!