1. (a) $a_n = 3(7^n)$. (b) $a_n = 2(5^n)$. (c) $a_n = 4(-1)^n$.

2. (a) The characteristic equation is $\lambda^2 - 5\lambda + 6 = (\lambda - 2)(\lambda - 3) = 0$ with roots 2 and 3, so the general solution is $a_n = C_1(2^n) + C_2(3^n)$ for some constants $C_1, C_2$. But

$$2 = a_0 = C_1 + C_2 \quad \text{and} \quad 5 = a_1 = 2C_1 + 3C_2.$$  

Solving yields $C_1 = C_2 = 1$, so that the final solution is

$$a_n = 2^n + 3^n.$$  

(b) The characteristic equation is $\lambda^2 - 4\lambda + 3 = (\lambda - 1)(\lambda - 3) = 0$ with roots 1 and 3, so the general solution is $a_n = C_1(1^n) + C_2(3^n)$ for some constants $C_1, C_2$. But

$$-1 = a_0 = C_1 + C_2 \quad \text{and} \quad 2 = a_1 = C_1 + 3C_2.$$  

Solving yields $C_1 = -5/2$ and $C_2 = 3/2$, so that the final solution is

$$a_n = \frac{1}{2}(3^{n+1} - 5).$$  

*(c) The characteristic equation is $\lambda^2 + 1 = 0$ with roots $i$ and $-i$, where $i = \sqrt{-1}$, so the general solution is $a_n = C_1(i^n) + C_2(-i)^n$ for some constants $C_1, C_2$. But

$$4 = a_0 = C_1 + C_2 \quad \text{and} \quad 6 = a_1 = iC_1 - iC_2.$$  

Solving yields $C_1 = 2 - 3i$ and $C_2 = 2 + 3i$, so that the final solution is

$$a_n = (2 - 3i)i^n + (2 + 3i)(-i)^n.$$  

3. (a) The characteristic equation is $\lambda^2 - 4\lambda + 4 = (\lambda - 2)^2 = 0$ with single root 2, so the general solution is $a_n = C_1(2^n) + C_2n(2^n)$ for some constants $C_1, C_2$. But

$$1 = a_0 = C_1 \quad \text{and} \quad 4 = a_1 = 2C_1 + 2C_2,$$

yielding $C_1 = C_2 = 1$, so that the final solution is

$$a_n = 2^n(1 + n).$$
(b) The characteristic equation is \( \lambda^2 - 6\lambda + 9 = (\lambda - 3)^2 = 0 \) with single root 3, so the general solution is \( a_n = C_1(3^n) + C_2n(3^n) \) for some constants \( C_1 \), \( C_2 \). But

\[
2 = a_0 = C_1 \quad \text{and} \quad -3 = a_1 = 3C_1 + 3C_2,
\]
yielding \( C_1 = 2 \) and \( C_2 = -3 \), so that the final solution is

\[
a_n = 3^n(2 - 3n).
\]

**(c) The characteristic equation is \( \lambda^3 - 6\lambda^2 + 12\lambda - 8 = (\lambda - 2)^3 = 0 \) with single root 2, so the general solution is

\[
a_n = C_12^n + C_2n2^n + C_3n^22^n
\]
for some constants \( C_1 \), \( C_2 \), \( C_3 \). But

\[
2 = a_0 = C_1, \quad 4 = a_1 = 2C_1 + 2C_2 + 2C_3, \quad 16 = 4C_1 + 8C_2 + 16C_3,
\]
yielding \( C_1 = 2 \), \( C_2 = -1 \) and \( C_3 = 1 \), so that the final solution is

\[
a_n = 2^n(2 - n + n^2).
\]

4. (a) The initial condition is that \( a_0 = 100 \), and the recurrence is

\[
a_{n+1} = \frac{201}{200}a_n + 10 \quad \text{for } n \geq 0.
\]

*(b) There are several ways to get a formula. For example, one could plug values into the formula in 1(c) of Tutorial 1. A direct calculation is as follows:

\[
a_n = \left(\frac{201}{200}\right)^n 100 + 10\left(\left(\frac{201}{200}\right)^{n-1} + \ldots + \left(\frac{201}{200}\right)^2 + \frac{201}{200} + 1\right)
\]

\[
= \left(\frac{201}{200}\right)^n 100 + 10\left(\frac{1 - \left(\frac{201}{200}\right)^n}{1 - \frac{201}{200}}\right)
\]

\[
= \left(\frac{201}{200}\right)^n 100 + \left(\frac{201}{200}\right)^n 2,000 - 2,000
\]

which simplifies to

\[
a_n = 2,100\left(\frac{201}{200}\right)^n - 2,000.
\]

*Check by induction: \( 100 = 2,100\left(\frac{201}{200}\right)^0 - 2,000 \) starts the induction.

Using the inductive hypothesis (when \( n = k \)), we get

\[
a_{k+1} = \frac{201}{200}a_k + 10 = \frac{201}{200}\left(2,100\left(\frac{201}{200}\right)^k - 2,000\right) + 10
\]

\[
= 2,100\left(\frac{201}{200}\right)^{k+1} - 2010 + 10 = 2,100\left(\frac{201}{200}\right)^{k+1} - 2,000
\]
which verifies the inductive step and hence also the general formula.

(c) After 10 and 20 years the account will contain $1,820.73 and $4,951.43 respectively. To become a millionaire we require $n$ years where $a_{12n} = 10^6$ so that

$$n = \frac{1}{12} \ln \left( \frac{1.002,000}{2.100} \right) = 103.05$$

so that the account will exceed $1,000,000 in the first month of the 104th year.

5. (a) After one hour has elapsed there will be $2^{12} = 4,096$ organisms. To get one million organisms we need $5n$ minutes where $2^n \geq 10^6$, so that $n \geq 20$, or at least 100 minutes. To get 100 million organisms we need $5n$ minutes where $2^n \geq 10^8$, so that $n \geq 27$, or at least 2 hours and 15 minutes.

*(b) A general formula is

$$a_n = \begin{cases} 2^n & \text{if } n \leq 11 \\ 2^n - 2,000(2^{n-11} - 1) & \text{if } n \geq 12 \end{cases}$$

The food remains safe provided

$$2^n - 2,000(2^{n-11} - 1) = 2^n \frac{48}{2,048} + 2,000 \leq 1,000,000$$

that is,

$$n \leq \log_2 \left( \frac{998,000 \times 2048}{48} \right) = 25.3$$

or, for 2 hours and 5 minutes after the fly landed (an extra 25 minutes compared with the situation where no antibiotic had been activated).

6. (a) $a_2 = 2^2, a_3 = 2^3, a_4 = 2^5, a_5 = 2^8, a_6 = 2^{13}$.

*(b) The induction begins because $a_0 = 2^1 = 2^{b_0}$ and $a_1 = 2^1 = 2^{b_1}$. Suppose as inductive hypothesis that $k \geq 2$ and $a_k = 2^{b_k}$ and $a_{k-1} = 2^{b_{k-1}}$. Then, using the fact that $b_{k+1} = b_k + b_{k-1}$,

$$a_{k+1} = a_k \cdot a_{k-1} = 2^{b_k} \cdot 2^{b_{k-1}} = 2^{b_k + b_{k-1}} = 2^{b_{k+1}}$$

establishing the inductive step and completing the proof.

**7. We claim that the Fibonacci number $b_n$ is even if and only if $n$ is one less than a multiple of 3. Here is a proof:

Certainly $b_0$ and $b_1$ are odd and neither 0 nor 1 is one less than a multiple of 3, which starts the induction. Suppose, as inductive hypothesis, that $k \geq 2$ and the claim holds for $n = k$ and $n = k - 1$. Recall that

$$b_{k+1} = b_k + b_{k-1} .$$
Case (i): If \( k = 3\ell \) for some \( \ell \) then \( k + 1 \) is not one less than a multiple of 3 and, by the inductive hypothesis, \( b_k \) is odd and \( b_{k-1} \) is even, so that \( b_{k+1} \) is odd.

Case (ii): If \( k = 3\ell + 2 \) for some \( \ell \) then again \( k + 1 \) is not one less than a multiple of 3 and, by the inductive hypothesis, \( b_k \) is even and \( b_{k-1} \) is odd, so that \( b_{k+1} \) is even.

Case (iii): If \( k = 3\ell + 1 \) for some \( \ell \) then this time \( k + 1 \) is one less than a multiple of 3 and, by the inductive hypothesis, both \( b_k \) and \( b_{k-1} \) are odd, so that \( b_{k+1} \) is even.

These three cases exhaust all of the possibilities and establish the claim for \( n = k + 1 \). Thus by induction the claim holds for all Fibonacci numbers.

8. We have \( a_0 = 1 \) and \( a_n = 2a_{n-1} \) if \( n \geq 0 \). Clearly \( a_n = 2^n \).

9. We have \( a_0 = 1 \), \( a_1 = 2 \) and \( a_n = a_{n-1} + a_{n-2} \) if \( n \geq 2 \). To see this notice that in a line of \( n \) people with \( n \geq 2 \), if the last person is \( F \) then there are \( a_{n-1} \) possibilities for the line of the first \( n-1 \) people, whilst if the last person is \( M \) then the second last person must be \( F \), so that there are \( a_{n-2} \) possibilities for the line of the first \( n-2 \) people. We get the Fibonacci sequence with the first element deleted.

**10.** (a) The induction begins because

\[
A_0 = \frac{1}{2} = -\frac{1}{14} \left( \frac{3}{10} \right)^0 + \frac{4}{7}, \quad B_0 = \frac{1}{2} = \frac{1}{14} \left( \frac{3}{10} \right)^0 + \frac{3}{7}.
\]

Suppose the claim holds for \( n = k \geq 0 \). Then

\[
A_{k+1} = \frac{7}{10}A_k + \frac{4}{10}B_k
= \frac{7}{10} \left( -\frac{1}{14} \left( \frac{3}{10} \right)^k + \frac{4}{7} \right) + \frac{4}{10} \left( \frac{1}{14} \left( \frac{3}{10} \right)^k + \frac{3}{7} \right)
= -\frac{1}{14} \left( \frac{3}{10} \right)^k \left( \frac{7}{10} - \frac{4}{10} \right) + \frac{28}{70} + \frac{12}{70}
= -\frac{1}{14} \left( \frac{3}{10} \right)^{k+1} + \frac{4}{7}
\]

and

\[
B_{k+1} = \frac{3}{10}B_k + \frac{6}{10}B_k
= \frac{3}{10} \left( -\frac{1}{14} \left( \frac{3}{10} \right)^k + \frac{4}{7} \right) + \frac{6}{10} \left( \frac{1}{14} \left( \frac{3}{10} \right)^k + \frac{3}{7} \right)
= \frac{1}{14} \left( \frac{3}{10} \right)^k \left( \frac{6}{10} - \frac{3}{10} \right) + \frac{12}{70} + \frac{18}{70}
= \frac{1}{14} \left( \frac{3}{10} \right)^{k+1} + \frac{3}{7}.
\]
which establishes the inductive step, and the result.

(b) In the long term (as \( n \to \infty \)), company \( A \) gets \( \frac{4}{7} \) and company \( B \) gets \( \frac{3}{7} \) of the market share.

(c) We have

\[
A_{n+1} = \frac{7}{10} A_n + \frac{4}{10} B_n , \quad \text{and} \quad B_{n+1} = \frac{3}{10} A_n + \frac{6}{10} B_n ,
\]

so that

\[
B_n = \frac{10}{4} \left( A_{n+1} - \frac{7}{10} A_n \right)
\]

and

\[
A_{n+2} = \frac{7}{10} A_{n+1} + \frac{4}{10} B_{n+1} \\
= \frac{7}{10} A_{n+1} + \frac{4}{10} \left( \frac{3}{10} A_n + \frac{6}{10} B_n \right) \\
= \frac{7}{10} A_{n+1} + \frac{4}{10} \left( \frac{3}{10} A_n + \frac{6}{10} \left( \frac{10}{4} \left( A_{n+1} - \frac{7}{10} A_n \right) \right) \right) \\
= \frac{13}{10} A_{n+1} - \frac{3}{10} A_n
\]

yielding

\[
A_{n+1} - \frac{13}{10} A_{n+1} + \frac{3}{10} A_n = 0 .
\]

Its characteristic equation is

\[
\lambda^2 - \frac{13}{10} \lambda + \frac{3}{10} = (\lambda - 1)(\lambda - \frac{3}{10}) = 0
\]

with roots 1 and \( \frac{3}{10} \). Thus, for some constants \( C_1 \) and \( C_2 \),

\[
A_n = C_1 + C_2 \left( \frac{3}{10} \right)^n
\]

and

\[
B_n = \frac{10}{4} \left( C_1 + C_2 \left( \frac{3}{10} \right)^{n+1} - \frac{7}{10} \left( C_1 + C_2 \left( \frac{3}{10} \right)^n \right) \right) \\
= \frac{10}{4} \left( \frac{3}{10} C_1 + C_2 \left( \frac{3}{10} \right)^n \left( \frac{3}{10} - \frac{7}{10} \right) \right) \\
= \frac{3}{4} C_1 - \left( \frac{3}{10} \right)^n C_2 .
\]

But

\[
\frac{1}{2} = A_0 = C_1 + C_2 \quad \text{and} \quad \frac{1}{2} = B_0 = \frac{3}{4} C_1 - C_2 .
\]
Solving gives
\[ C_1 = \frac{4}{7} \quad \text{and} \quad C_2 = -\frac{1}{14} \]
from which the formulae in (a) quickly follow.

(d) Suppose more generally that the initial market shares are \( A_0 \) and \( B_0 \). Notice that \( A_0 + B_0 = 1 \). In the final part of the derivation of part (c) we get instead
\[ A_0 = C_1 + C_2 \quad \text{and} \quad B_0 = \frac{3}{4}C_1 - C_2 \]
which have solutions
\[ C_1 = \frac{4}{7} \quad \text{and} \quad C_2 = A_0 - \frac{4}{7}. \]
Thus
\[ A_n = \frac{4}{7} + \left(A_0 - \frac{4}{7}\right)\left(\frac{3}{10}\right)^n \rightarrow \frac{4}{7} \quad \text{as} \quad n \rightarrow \infty \]
and
\[ B_n = \frac{3}{4}\left(\frac{4}{7}\right) - \left(A_0 - \frac{4}{7}\right)\left(\frac{3}{10}\right)^n \rightarrow \frac{3}{7} \quad \text{as} \quad n \rightarrow \infty. \]
Thus the long term behaviour is the same for all possible initial market shares.