Starred questions are suitable for students aiming for a credit or higher.

1. This problem solves a nonhomogeneous recurrence in stages.
   (a) Solve the homogeneous recurrence $a_n = 3a_{n-1} + 4a_{n-2}$.
   (b) Find a particular solution $p_n = An + B$ of the recurrence
       
       \[ a_n = 3a_{n-1} + 4a_{n-2} - 12n - 2 \]
       
       and use part (a) to write down the general solution.
   (c) Solve the recurrence in (b) given also that $a_0 = 2, a_1 = 3$.

2. Use the method of the previous problem to solve the following recurrences:
   (a) $a_n = 5a_{n-1} - 6a_{n-2} + 2n + 3$ for $n \geq 2$ where $a_0 = 2, a_1 = 5$.
   *(b) $a_n = 5a_{n-1} - 6a_{n-2} + 4^n + 2n + 3$ for $n \geq 2$ where $a_0 = 5, a_1 = 19$.
   (c) $a_n = 4a_{n-1} - 4a_{n-2} + 2$ for $n \geq 2$ where $a_0 = -1, a_1 = 2$.
   *(d) $a_n = 4a_{n-1} - 4a_{n-2} + 3^n - 6n + 5$ for $n \geq 2$ where $a_0 = 0, a_1 = -2$.
   **(e) $a_n = 4a_{n-1} - 4a_{n-2} + 2^n$ for $n \geq 2$ where $a_0 = a_1 = 1$.

3. Find simple expressions for the generating functions of the following sequences:
   (a) $1, 3, 3^2, 3^3, \ldots$  
   (b) $1, -1, 1, -1, \ldots$  
   (c) $2, 2, 2, \ldots$  
   *(d) $1, -2, 3, -4, \ldots$

4. Write down the sequences associated with the following generating functions:
   (a) $(3 + 4z)^2$  
   (b) $\frac{1}{1 + 2z}$  
   (c) $\frac{z}{1-z}$  
   **(d) $\frac{3z - 1}{(1+z)^2}$

5. Let $G(z) = \sum_{n=0}^{\infty} a_n z^n$. Express in terms of $G(z)$ the generating functions for
   (a) $b_n = 10a_n$  
   (b) $c_n = a_n - 6$  
   *(c) $d_n = a_{n-2}$  
   **(d) $e_n = a_{n+2}$

6. Let $G(z)$ be the generating function of the sequence defined by $a_0 = 5$ and
   $a_n = 2a_{n-1} + 1$ for $n \geq 1$. Verify that
   
   \[ G(z)(1 - 2z) = \frac{5 - 4z}{1 - z} \]
   
   and use partial fractions and geometric series to deduce the formula
   
   \[ a_n = 6(2^n) - 1. \]
**7.** Let $G(z)$ be the generating function for the sequence defined by the recurrence
\[ a_n = \frac{1}{2}a_{n-1} + a_{n-2} - \frac{1}{2}a_{n-3} \]
where $a_0 = 8$, $a_1 = 12$, $a_2 = 5$. Verify that
\[ G(z) = \frac{8 + 8z - 9z^2}{(1-z)(1+z)(1-\frac{1}{2}z)} \]
and find a formula for $a_n$. What happens to $a_{2n}$ and $a_{2n+1}$ as $n \to \infty$?

**8.** Verify that
\[ \frac{z(1+z)}{(1-z)^3} = \sum_{n=0}^{\infty} n^2 z^n . \]

**9.** Let $G(z) = \sum_{n=0}^{\infty} a_n z^n$ and $H(z) = \sum_{n=0}^{\infty} (a_0 + a_1 + \cdots + a_n) z^n$. Verify that
\[ H(z) = \frac{G(z)}{1-z} \]
and use the previous exercise to discover a formula for
\[ 1^2 + 2^2 + \cdots + n^2 . \]

**10.** In this problem we use generating functions to solve 10(d) of Tutorial 2, concerning long term market shares of companies $A$ and $B$, where we assume $A_0 + B_0 = 1$.

(a) Suppose $G(z) = A_0 + A_1 z + A_2 z^2 + \cdots$ and $H(z) = B_0 + B_1 z + B_2 z^2 + \cdots$ where
\[ A_{n+1} = \frac{7}{10} A_n + \frac{4}{10} B_n \quad \text{and} \quad B_{n+1} = \frac{3}{10} A_n + \frac{6}{10} B_n \]
for $n \geq 0$. Verify that
\[ A_0 = \left( 1 - \frac{7}{10} z \right) G(z) - \frac{4}{10} z H(z) \]
and
\[ B_0 = -\frac{3}{10} z G(z) + \left( 1 - \frac{6}{10} z \right) H(z) . \]

(b) Solve these equations simultaneously for $G(z)$ and $H(z)$.

(c) Use partial fractions and geometric series to deduce the formulae
\[ A_n = \frac{4}{7} + \left( \frac{3}{7} - B_0 \right) \left( \frac{3}{10} \right)^n \]
and
\[ B_n = \frac{3}{7} + \left( \frac{4}{7} - A_0 \right) \left( \frac{3}{10} \right)^n . \]

(d) Write down $\lim_{n \to \infty} A_n$ and $\lim_{n \to \infty} B_n$ and notice that the answers are independent of $A_0$ and $B_0$. 