The University of Sydney
MATH2011 Topics in Discrete Mathematics

Semester 1 Tutorial 6 2004

Starred questions are suitable for students aiming for a distinction or higher.

1. Let $X = 111110$ and $Y = 110111$ be integers written in base 2.
   (a) To which integers do $X$ and $Y$ correspond in base 10?
   (b) Find $XY$ using traditional long multiplication in base 2 arithmetic. What does your answer correspond to in base 10?
   (c) Observe that $X = 10^3A + B$ and $Y = 10^3B + A$ where $A = 111$ and $B = 110$. Find $AB$ and $(A - B)^2$ working in base 2. Now find $10^6AB + 10^3((A - B)^2 + 10AB) + AB$.
   (d) Can you explain why you get the same answer as (b)? (This is an example of the divide-and-conquer technique. The calculations $AB$ and $(A - B)^2$ and the merged formula in (c) are intended to be briefer than (b).)

2. Here is a recursive divide-and-conquer algorithm for multiplication of integers in base 2, which has faster running time than the traditional $O(N^2)$ long multiplication. We input integers $X, Y$ in base 2, each assumed to have $N = 2^k$ digits. (We can always guarantee this by padding with zeros if necessary.)

   MULTIPLY$(X, Y)$:
   (1) (trivial step) If $N = 1$ then output 1 if $X = Y = 1$; output 0 if $X = 0$ or $Y = 0$; output $-1$ otherwise; and stop.
   (2) (divide step) Write $X = 10^{N/2}A + B$ and $Y = 10^{N/2}C + D$.
   (3) (conquer step) Find
      $P := \text{output of } \text{MULTIPLY}(A, C)$
      $Q := \text{output of } \text{MULTIPLY}(B, D)$
      $R := \text{output of } \text{MULTIPLY}(A - B, D - C)$
   (4) (merge step) Output $10^NP + 10^N/2(P + Q + R) + Q$ and stop.

   (a) Explain why the output is the product $XY$ in base 2.
   *(b) Explain briefly why the running time is

   $$T(N) = \begin{cases} 
   O(1) & \text{for } N = 1 \\
   3T(N/2) + O(N) & \text{for } N > 1.
   \end{cases}$$

   **(c) Solve the previous recurrence to verify that

   $$T(N) = O(N^{\log_2 3}) \approx O(N^{1.585})$$.
3. Let \( P(x_1, y_1), Q(x_2, y_2), R(x_3, y_3) \) be points in the \( xy \)-plane, and consider the vectors \( \mathbf{u} = \overrightarrow{PQ} \) and \( \mathbf{v} = \overrightarrow{PR} \).

(a) Put

\[
\Delta_{PQR} = \det \begin{bmatrix} 1 & 1 & 1 \\ x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \end{bmatrix}
\]

and verify that

\[
\Delta_{PQR} = (x_2 - x_1)(y_3 - y_1) - (x_3 - x_1)(y_2 - y_1).
\]

(b) Find the cross product \( \mathbf{u} \times \mathbf{v} \) where \( \mathbf{u} \) and \( \mathbf{v} \) are regarded as vectors in space, and compare your answer to part (a).

(c) Explain why the triangle \( \triangle PQR \) is oriented anticlockwise if \( \Delta_{PQR} > 0 \) and clockwise if \( \Delta_{PQR} < 0 \). What happens if \( \Delta_{PQR} = 0 \)?

(d) Let \( P = (5, 1), Q = (7, 9) \) and \( R = (1, 4) \). What is \( \Delta_{PQR} \)? In each case below, find \( \Delta_{PQS}, \Delta_{PSR} \) and \( \Delta_{SQR} \) and deduce whether \( S \) lies inside or outside the triangle \( \triangle PQR \):

(i) \( S = (3, 3) \)  
(ii) \( S = (4, 7) \)  
(iii) \( S = (6, 5) \)

4. Denote by \( P_0 \) the origin of the \( xy \)-plane. The polar angle of a point \( P \) in the plane is the anticlockwise angle the vector \( \overrightarrow{P_0P} \) makes with the positive \( x \)-axis.

(a) Put the following fractions in decreasing order:

\[
\frac{3}{6}, \frac{1}{3.5}, \frac{0.5}{1}, \frac{22}{23}.
\]

Now order the following points by increasing polar angle:

\[
P_1(3, 6), \ P_2(1, 1), \ P_3(0.5, 3.5), \ P_4(0, 1), \ P_5(22, 23).
\]

*(b) Suppose \( P_1, \ldots, P_N \) are points in the first quadrant of the \( xy \)-plane excluding \( P_0 \) which have been sorted by increasing polar angle. Suppose further that all of the angles are different. Using \( \Delta_{P_i P_{i+1} P_{i+2}} \) for \( i = 1 \) to \( N-1 \), devise a simple test with \( O(N) \) running time which decides whether \( P_0, P_1, \ldots, P_N \) are vertices of a convex polygon.

*(c) Apply your test in (b) to the sorted list from (a).

**(d) In part (b), the points \( P_0, P_1, \ldots, P_N \) are always vertices of a simple polygon. Give a test with \( O(\log N) \) running time which decides whether any given point \( Q \) in the plane lies inside or outside this simple polygon.

**(e) Decide whether \( Q(15, 20) \) lies inside or outside the polygon for the sorted list of points from (a).