Predicate Calculus: deduction rules

Recall 10 deduction rules for the Propositional Calculus:

\[ \text{A, MP, MT, DN, CP, \land I, \land E, \lor I, \lor E, RAA} \]

We have 4 further rules to make up the Predicate Calculus:

\[ \forall E, \forall I, \exists I, \exists E \]

- analogues of \( \land E, \land I, \lor I, \lor E \) when we think of the statements as gigantic conjunctions and \exists \) statements as gigantic disjunctions.

**Universal Elimination (UE):** From \( (\forall x) S(x) \) deduce \( S(a) \) for any constant symbol uniformly substituted for \( x \) throughout.
Universal Introduction (UI): From \( S(a) \) deduce
\((\forall x)\ S(x)\), uniformly substituting \( x \) for \( a \) throughout provided \( a \) does not appear in any underlying assumptions on which \( S(a) \) depends.

Example (syllogism):

- All men are mortal.
- Socrates is a man.

\[ \therefore \text{ Socrates is mortal.} \]

Let \( h(x) \) denote "\( x \) is a man" and \( m(x) \) denote "\( x \) is mortal". The syllogism becomes

\[ h(a), (\forall x) (h(x) \Rightarrow m(x)) \Rightarrow m(a) \]

Proof in the predicate calculus:

\[
\begin{array}{c}
1 & (1) & h(a) & A \\
2 & (2) & (\forall x) (h(x) \Rightarrow m(x)) & A \\
2 & (3) & h(a) \Rightarrow m(a) & 2 \text{ VE} \\
1, 2 & (4) & m(a) & 1, 3 \text{ MP} \\
\end{array}
\]
Main idea in using $\forall E$ and $\forall I$:

- Universally quantified statements are like "boxes" that we open up (using $\forall E$) and access the "content". 
- Under appropriate circumstances (using $\forall I$) we can create "new boxes" and close them up.

Example: Prove

$$(\forall x) \ (F(x) \rightarrow G(x)) \quad (\forall x) \ F(x) \vdash (\forall x) \ G(x)$$

Proof:
1. $$(\forall x) \ F(x) \rightarrow G(x) \quad 1$$
2. $$(\forall x) \ F(x) \quad 2 \ \forall I$$
   - $$(\forall x) \ F(x) \rightarrow G(x)$$
   - $$F(a) \rightarrow G(a)$$
   - $$G(a)$$

Justified because a dot not appear in (1) or (2).

Incorrect use of $\forall I$:

1. $$(\forall x) \ M(x) \quad 1 \forall E$$
2. $$(\forall x) \ M(x) \quad 1 \forall I$$
   - $$(\forall x) \ M(x) \quad 1 \forall I$$

$a$ is being replaced by "$x$" but appear in (1) in which (1) useless.
The previous incorrect use of $\forall I$ would be like trying to reason

"Some person is Greek"  
\[ \forall \text{ people are Greek} \]

However, it is possible and valid to reason

"Some person is Greek"  
\[ \therefore \text{ some person is Greek} \]

Using $\exists I$:

\[
\therefore (\exists x) \, \phi(x) \quad ! \exists I
\]

Extended Introduction ($\exists I$):

Given a statement $\phi(a)$ about an object $a$ in $\mathcal{U}$, we may conclude

\[
\exists x \, \phi(x)
\]

Substituting at least one occurrence of $a$ by $x$ (not necessarily all occurrences of $a$).
There are no restrictions on underlying assumptions for $S(x)$, and these are carried through to $(\exists \alpha) S(\alpha)$, noted as usual on the left.

Example: prove

\[
(\forall x) A(x) \quad \vDash \quad (\exists x) A(x)
\]

Proof:
1. (1) \((\forall x) A(x)\) \ A
2. (2) \(A(\alpha)\) \ I \ A
3. (3) \((\exists x) A(x)\) \ E \ I

Note: (4) would fail to make semantic sense (and the conclusion false) if \(U = \emptyset\), for the existence of anything in \(U\) would fail, having property \(A\) or any other property, while \((\forall x) A(x)\) becomes vacuously true, being an implication in disguise.

"If \(x \in U\) then \(A(x)\) holds!"
Example: Prove

\[(\forall x) \ E(x, x) \iff (\exists y) \ (\forall x) \ E(x, y)\]

\[\text{and}\]

\[(\forall x) \ E(x, x) \iff (\exists y) \ (\exists x) \ E(x, y)\]

Part 1: \(\forall x\) : 

1. \(\forall x) \ E(x, x) \ A\)
2. \((\exists y) \ E(x, y) \ 1 \ SI (5) (\sim)\)

Note: In the substitution, we are forming a predicate \(G\) out of the binary relation \(E\) by \(G(m) = E(x, x)\)

Part 1: \(\exists x\) : 

1. \((\forall x) \ E(x, x) \ A\)
2. \(E(a, a) \ 1 \ VE\)
3. \((\exists y) \ E(x, y) \ 2 \ EI\)
4. \((\exists y) (\exists x) \ E(x, y) \ 3 \ EI\)

Note: With \(\exists x, \exists y\), the variable introduced needs to be new.
We motivate the final rule, Excluded Element Elimination (EE), by the following:

Example: Prove

\[(\forall x)(F(x) \rightarrow G(x)), (\exists x)F(x) \rightarrow (\exists x)G(x)\]

Proof:

1. (1) \((\forall x)F(x) \Rightarrow G(x)\)
2. (2) \((\exists x)F(x)\)
3. (3) \(F(a)\)
   1. (4) \(F(a) \Rightarrow G(a)\)
   2. (5) \(G(a)\)
   3. (6) \((\exists x)G(x)\)
   4. (7) \((\forall x)G(x)\)
   5. (8) \((\forall x)G(x) \rightarrow G(a)\)
   6. (9) \(G(a)\)

\[\therefore (\exists x)G(x)\]

- Temporary assumption in typing a "different" from \(a\)
- \(3\) is discharged in favour of \(2\)
- Only 3 lines
- Compared with 5 lines for VE
Existential Elimination (EE):

Given \((\exists x) S(x)\) as a premise and \(S(a)\)

invoked by Rule of Assumption (A) and \(a\) is a constant symbol (representing an actual object in \(\mathcal{U}\))

replacing \(x\) uniformly throughout, and given a conclusion \(C\) (possibly complicated)

relying on \(S(a)\), we may rewrite

\[ C \]

again, but relying on \((\exists x) S(x)\) instead

(discharging the assumption \(S(a)\), pooling any assumptions for \((\exists x) S(x)\) and any assumptions used to derive \(C\) from \(S(a)\), provided \(a\) does not appear in \(C\) or any underlying assumptions

When invoking EE, only 3 lines are quoted, one for \((\exists x) S(x)\)
on one for \(S(a)\) invoked by (A), and one for line where \(C\) occurs.