Fourier Transforms

Recall

The complex Fourier series of \( f(x) \) is given by:

\[
f(x) = \sum_{n=-\infty}^{\infty} c_n e^{\frac{in\pi x}{L}}
\]

where

\[
c_n = \frac{1}{2L} \int_{-L}^{L} f(k)e^{\frac{in\pi k}{L}} \, dk
\]
Fourier transforms

Substitute the integral for the coefficient into the sum,

\[
f(x) = \sum_{n=-\infty}^{\infty} \left[ \frac{1}{2L} \int_{-L}^{L} f(k) e^{i\pi k} \, dk \right] e^{i\pi x} \frac{1}{L}
\]

Introduce the variable \( \omega_n = \frac{n\pi}{L} \)

Then

\[
\frac{\Delta \omega}{2\pi} = \frac{\omega_{n+1}}{2\pi} - \frac{\omega_n}{2\pi} = \frac{1}{2L}
\]

\[
f(x) = \sum_{n=-\infty}^{\infty} \left[ \frac{1}{2\pi} \int_{-\infty}^{\infty} f(k) e^{i\omega_n k} \, dk \right] e^{-i\omega_n x} \Delta \omega
\]

Fourier transforms

Consider now the limit as \( L \to \infty \), then \( \Delta \omega \equiv \frac{\pi}{L} \to 0 \)

We can therefore interpret the sum as a Riemann sum and in the limit it is replaced by an integral with respect to the continuous variable \( \omega \)

I.e.,

\[
f(x) = \int_{-\infty}^{\infty} \left[ \frac{1}{2\pi} \int_{-\infty}^{\infty} f(k) e^{i\omega k} \, dk \right] e^{-i\omega x} \, d\omega
\]

This is the Fourier Integral Identity
**Fourier transforms**

We now define the *Fourier Transform* of \( f(x) \) as

\[
F(\omega) = \frac{1}{2\pi} \int_{-\infty}^{\infty} f(x) e^{i\omega x} \, dx
\]

Note the change of dummy variable from \( k \) to \( x \).

Then, from the integral identity we define the *Inverse Transform*

\[
f(x) = \int_{-\infty}^{\infty} F(\omega) e^{-i\omega x} \, d\omega
\]

**Existence:** \( f(x) \) must be piecewise smooth and absolutely integrable:

\[
\int_{-\infty}^{\infty} |f(x)| \, dx < \infty
\]

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**Fourier transforms**

Alternative notation

\[
F(\omega) = \mathcal{F}\{f(x)\}
\]

\[
f(x) = \mathcal{F}^{-1}\{F(\omega)\}
\]

Important:

There are variations in the definition of the Fourier transform and its inverse, especially in the placement of the \( \frac{1}{2\pi} \) factor and the sign of the complex exponential.
Inverse Fourier transform of a Gaussian

To find the solution of the heat equation and other problems, we will need to find the inverse Fourier transform of the function

\[ G(\omega) = e^{-\alpha \omega^2} \]

This is the well-known bell-shaped curve known as a Gaussian.

By definition, the inverse transform is given by

\[ g(x) = \int_{-\infty}^{\infty} G(\omega) e^{-i\omega x} \, d\omega \]

\[ = \int_{-\infty}^{\infty} e^{-\alpha \omega^2} e^{-i\omega x} \, d\omega \]

Inverse Fourier transform of a Gaussian

To evaluate this integral we use the following “trick”: First, differentiate with respect to \( x \)

\[ g'(x) = \int_{-\infty}^{\infty} (-i\omega) e^{-\alpha \omega^2} e^{-i\omega x} \, d\omega \]

Next, integrate by parts:

\[ g'(x) = \frac{i}{2\alpha} \int_{-\infty}^{\infty} \frac{d}{d\omega} (e^{-\alpha \omega^2}) e^{-i\omega x} \, d\omega \]

\[ = \frac{i}{2\alpha} \left[ e^{-i\omega x} e^{-\alpha \omega^2} \right]_{-\infty}^{\infty} \]

\[ - \frac{i}{2\alpha} \int_{-\infty}^{\infty} e^{-\alpha \omega^2} (-ix) e^{-i\omega x} \, d\omega \]
Inverse Fourier transform of a Gaussian

The first term vanishes as \( \omega \to \pm \infty \)

Therefore

\[
g'(x) = -\frac{x}{2\alpha} \int_{-\infty}^{\infty} e^{-\alpha \omega^2} e^{-i\omega x} \, d\omega
\]

\[
\text{I.e.,} \quad g'(x) = -\frac{x}{2\alpha} g(x)
\]

This is a simple separable first order ODE for \( g(x) \).

\[
\frac{g'(x)}{g(x)} = -\frac{x}{2\alpha} \quad \Rightarrow \quad g(x) = g(0)e^{-x^2/4\alpha}
\]

Inverse Fourier transform of a Gaussian

But

\[
g(0) = \int_{-\infty}^{\infty} e^{-\alpha \omega^2} \, d\omega
\]

and it can be shown that

\[
\int_{-\infty}^{\infty} e^{-\alpha \omega^2} \, d\omega = \frac{\sqrt{\pi}}{\sqrt{\alpha}}
\]

Therefore, the final expression for the inverse transform of the Gaussian is:

\[
g(x) = \frac{\sqrt{\pi}}{\sqrt{\alpha}} e^{-x^2/4\alpha}
\]

which itself is another Gaussian.
The Dirac delta function

Let us consider the sequence of functions

\[ \delta_n(x) = \frac{\sqrt{n}}{\pi} e^{-nx^2}, \quad n = 1, 2, 3, \ldots \]

We can show that \[ \int_{-\infty}^{\infty} \frac{\sqrt{n}}{\pi} e^{-nx^2} \, dx = 1, \quad n = 1, 2, 3, \ldots \]

We define the Dirac delta function \( \delta(x) \) to be the limit of that sequence as \( n \to \infty \) such that

\[ \lim_{n \to \infty} \int_{-\infty}^{\infty} \frac{\sqrt{n}}{\pi} e^{-nx^2} \, dx = \int_{-\infty}^{\infty} \delta(x) \, dx = 1 \]

The Dirac delta function

We can think of the delta function as an infinitely concentrated pulse which is zero everywhere, except at \( x = x_0 \) where it is \( \infty \)

i.e.,

\[ \delta(x - x_0) = \begin{cases} 0, & x \neq x_0 \\ \infty, & x = x_0 \end{cases} \]

The Dirac delta function has a physical analogy of an "impulsive" force acting for a short time only.
The Dirac delta function - Properties

1) Filtering property:
\[ \int_{-\infty}^{\infty} \delta(x - x_0) f(x) \, dx = f(x_0) \]

2) Operates like an even function:
\[ \delta(x - x_0) = \delta(x_0 - x) \]

3) Derivative of Heaviside step function \( H(x - x_0) = \begin{cases} 0, & x < x_0 \\ 1, & x > x_0 \end{cases} \)
is the delta function:
\[ H'(x - x_0) = \delta(x - x_0) \]

This can be seen by realizing that
\[ \int_{-\infty}^{x} \delta(\lambda - x_0) \, d\lambda = \begin{cases} 0, & x < x_0 \\ 1, & x > x_0 \end{cases} \]

The Dirac delta function

4) Fourier transform of the delta function
\[ \mathcal{F}\{\delta(x - x_0)\} = \frac{1}{2\pi} \int_{-\infty}^{\infty} \delta(x - x_0) e^{i\omega x} \, dx = \frac{1}{2\pi} e^{i\omega x_0} \]

Alternatively:
\[ \mathcal{F}^{-1}\left\{ \frac{1}{2\pi} e^{i\omega x_0} \right\} = \delta(x - x_0) \]

I.e.,
\[ \int_{-\infty}^{\infty} e^{-i\omega(x-x_0)} \, d\omega = 2\pi \delta(x - x_0) \]

In the special case where \( x_0 = 0 \) we get
\[ \mathcal{F}\{\delta(x)\} = \frac{1}{2\pi} \text{ or } \int_{-\infty}^{\infty} e^{-i\omega x} \, d\omega = 2\pi \delta(x) \]
Derivatives and Convolution

5) Fourier transforms of derivatives

\[ \mathcal{F}\{f^{(n)}(x)\} = (-i\omega)^n \mathcal{F}\{f(x)\} \]

(Proof by integration by parts)

6) Convolution: if \( F(\omega) = \mathcal{F}\{f(x)\} \) and \( G(\omega) = \mathcal{F}\{g(x)\} \)
then

\[ \mathcal{F}^{-1}\{F(\omega)G(\omega)\} = f \ast g \]

where

\[ f \ast g = \frac{1}{2\pi} \int_{-\infty}^{\infty} g(\lambda)f(x-\lambda) \, d\lambda \]

Is the convolution of \( f \) and \( g \).

Heat equation using Fourier transforms

Use Fourier transforms to solve the 1-D Heat equation

\[ \frac{\partial u}{\partial t} - k \frac{\partial^2 u}{\partial x^2} = 0, \quad -\infty < x < \infty, \quad t > 0 \]

with initial condition \( u(x, 0) = f(x) \)

There are implicit physical conditions such as

\[ u(x, t) \to 0 \text{ as } x \to \pm \infty \]

Take the FT with respect to \( x \).

Define

\[ u(\omega, t) = \mathcal{F}\{u(x, t)\} \equiv \frac{1}{2\pi} \int_{-\infty}^{\infty} u(x, t)e^{i\omega x} \, dx \]
Heat equation using Fourier transforms

Then
\[ \mathcal{F} \left\{ \frac{\partial u}{\partial t} \right\} = \frac{\partial U(\omega, t)}{\partial t} \quad \mathcal{F} \left\{ \frac{\partial^2 u}{\partial x^2} \right\} = (-i\omega)^2 U(\omega, t) \]

Substitute into the PDE \[ \frac{\partial U}{\partial t} + k\omega^2 U = 0 \]

Solving the ODE \[ U(\omega, t) = C(\omega) e^{-k\omega^2 t} \] (*)

Inverting \[ u(x, t) = \int_{-\infty}^{\infty} C(\omega) e^{-k\omega^2 t} e^{-i\omega x} d\omega \]

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Heat equation using Fourier transforms

Letting \( t = 0 \) \[ u(x, 0) = \int_{-\infty}^{\infty} C(\omega) e^{-i\omega x} d\omega \]

Then, the initial condition \( u(x, 0) = f(x) \) gives

\[ f(x) = \int_{-\infty}^{\infty} C(\omega) e^{-i\omega x} d\omega \]

from which we can calculate \( C(\omega) \) by inversion.

Ie,

\[ C(\omega) = \frac{1}{2\pi} \int_{-\infty}^{\infty} f(x) e^{i\omega x} dx \equiv \mathcal{F}[f(x)] \]
Heat equation using Fourier transforms

Summarising, the solution of the heat equation in the infinite domain $-\infty < x < \infty$ with IC $u(x,0) = f(x)$ is given by

$$u(x,t) = \int_{-\infty}^{\infty} C(\omega) e^{-k\omega^2 t} e^{-i\omega x} d\omega$$

where

$$C(\omega) = \mathcal{F}[f(x)]$$

We can now rewrite this as a convolution, but alternatively we can use (*) from the outset:

Recall that

$$U(\omega,t) = C(\omega)e^{-k\omega^2 t}$$

Hence, $u(x,t) = \mathcal{F}^{-1}\{C(\omega)e^{-k\omega^2 t}\}$

$$= \mathcal{F}^{-1}\{C(\omega)\} \ast \mathcal{F}^{-1}\{e^{-k\omega^2 t}\}$$

But

$$\mathcal{F}^{-1}\{C(\eta)\} = f(x)$$

and (from the inverse of a Gaussian):

$$\mathcal{F}^{-1}\{e^{-k\omega^2 t}\} = \frac{\sqrt{\pi}}{\sqrt{kt}} e^{-x^2/4kt}$$

Therefore

$$u(x,t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} f(\bar{x}) \sqrt{\frac{\pi}{kt}} e^{-\frac{(x-\bar{x})^2}{4kt}} d\bar{x}$$
Heat equation using Fourier transforms

In the special case where \( f(x) = \delta(x) \)

\[
\begin{align*}
    u(x, t) &= \frac{1}{2\pi} \int_{-\infty}^{\infty} \delta(\bar{x}) \left[ \frac{\pi}{\sqrt{kt}} e^{-\frac{(x-\bar{x})^2}{4kt}} \right] d\bar{x} \\
    &= \frac{1}{2\pi} \sqrt{\frac{\pi}{kt}} e^{-\frac{x^2}{4kt}}
\end{align*}
\]

This is called the **fundamental solution of the heat equation**

It is the response at time \( t \) and position \( x \) to an initial input concentrated at \( x = 0 \) and \( t = 0 \)