LAPLACE'S EQUATION FOR A CIRCULAR DISK

We use Laplace's eqn in POLAR coordinates

\[ \nabla^2 u = \frac{1}{r} \frac{\partial}{\partial r} \left( r \frac{\partial u}{\partial r} \right) + \frac{1}{r^2} \frac{\partial^2 u}{\partial \theta^2} = 0 \]

where \( x = r \cos \theta \), \( y = r \sin \theta \)

Find \( u = u(r, \theta) \) which satisfies \( \nabla^2 u = 0 \) in a DISK of radius \( a \),

\[ \text{B.C.} \quad u(a, \theta) = f(\theta) \]

For \( 0 \leq r \leq a \), \( -\pi \leq \theta \leq \pi \)

The solution must also satisfy (non-physical conditions) at \( r = 0 \) and \( \theta = \pm \pi \).
**BOUNDEDNESS**

a) \( u(r, \theta) \) must remain BOUNDED as \( r \to 0 \)

**PERIODICITY**

b) \( u(r, -\pi) = u(r, \pi) \) for \( 0 \leq r \leq a \)

c) \( \frac{\partial u}{\partial \theta}(r, -\pi) = \frac{\partial u}{\partial \theta}(r, \pi) \) \( 0 \leq r \leq a \)

**POSSIBLE APPLICATIONS**

1) **the STEADY-STATE**

   temperature distribution in a circular disk, with specified temperature distribution \( u(a, \theta) \) on the boundary.

2) **FLUID FLOW** past a circular cylinder

   We will sketch the solution to this problem in the next lecture.
SEPARATION OF VARIABLES

Let $u(r, \theta) = R(r) G(\theta)$ and substitute into Laplace’s eqn:

$$\frac{1}{r} \frac{\partial}{\partial r} \left[ r \frac{\partial}{\partial r} (RG) \right] + \frac{1}{r^2} \frac{\partial^2}{\partial \theta^2} (RG) = 0$$

$$G \frac{1}{r} \frac{d}{dr} (rR') + \frac{R}{r^2} G''' = 0$$

$$X \left( \frac{R}{RG} \right) \Rightarrow$$

$$\frac{R}{R} \frac{d}{dr} (rR') = - \frac{G''}{G} = K$$

$K = \text{SEPARATION CONSTANT}$

$\Rightarrow$ 2 ODEs

$$G'' + KG = 0 \quad (2) \quad G = G(\theta)$$

$$r \frac{d}{dr} (rR') - KR = 0 \quad R = R(r) \quad (3)$$
SOLVE THE Θ EQN

\[ C'' + KC = 0 \]

Recall that we want periodic BCs in θ \( \Rightarrow \) want periodic solutions \( \sin \theta \ & \ \cos \theta \)

\[ \Rightarrow K = \lambda^2 \ \ \lambda > 0 \]

\[ \Rightarrow C'' + \lambda^2 C = 0 \]

\[ \Rightarrow C(\theta) = A \sin \lambda \theta + B \cos \lambda \theta \]

A, B arbitrary constants

Apply the periodic BCs.

\[ u(v, -\pi) = u(v, \pi) \Rightarrow C(-\pi) = C(\pi) \]

\[ \frac{\partial u}{\partial \theta} (v, -\pi) = \frac{\partial u}{\partial \theta} (v, \pi) \Rightarrow \frac{\partial C}{\partial \theta}(-\pi) = \frac{\partial C}{\partial \theta}(\pi) \]
\[ g(-\pi) = A \cos \lambda \pi - B \sin \lambda \pi \]
\[ g(\pi) = A \cos \lambda \pi + B \sin \lambda \pi \]

\[ g(-\pi) = g(\pi) \implies 2B \sin \lambda \pi = 0 \]

Similarly
\[ g'(-\pi) = \lambda A \sin \lambda \pi + B \lambda \cos \lambda \pi \]
\[ g'(\pi) = -\lambda A \sin \lambda \pi + B \lambda \cos \lambda \pi \]

\[ g'(-\pi) = g'(\pi) \implies 2\lambda A \sin \lambda \pi = 0 \]

In both cases, for non-trivial solutions, we must have

\[ \sin \lambda \pi = 0 \]

\[ \implies \lambda = n, \ n=1, 2, 3, \ldots \]
NOTE that for $\lambda = 0$ the BCs are also satisfied and $\alpha(0) = A_0$ (arbitrary constant).

Therefore, the solution for $\alpha(\theta)$ is:

$$\alpha_n(\theta) = A_n \cos n\theta + B_n \sin n\theta$$

$$n = 1, 2, 3, \ldots$$

$$\alpha_0(\theta) = A_0 \quad n = 0.$$ 

Which can be combined as:

$$\alpha_n(\theta) = A_n \cos n\theta + B_n \sin n\theta$$

$$n = 0, 1, 2, \ldots$$
SOLVE THE \( R(r) \) EQN

Letting \( K = \lambda^2 = \frac{q}{h} \)
in eqn (3) gives

\[
2 \frac{d}{dr} (rR') - hR = 0
\]

Expanding \( \Rightarrow \)

\[
r^2 R'' + rR' - hR = 0 \tag{5}
\]

This is an Euler-Cauchy type ODE.

For \( h \neq 0 \) try:

\[
R = r^p
\]

\[
\Rightarrow R' = pr^{p-1} \quad \text{and} \quad R'' = p(p-1)r^{p-2}
\]

Substitute into (5):

\[
r^{p-2} p(p-1)r^{p-1} + rpr - hr = 0
\]

Dividing by \( r^p \) gives

\[
p(p-1) + p - h = 0
\]
\[ \Rightarrow \quad \rho^2 - n^2 = 0 \]
\[ \Rightarrow \quad \rho = \pm n \]
\[ \Rightarrow \quad R(r) = Ar^n + \frac{B}{r^n} \quad n \neq 0 \]

A, B arbitrary constants.

Since we must have
\[ u(r, \theta) \text{ BOUNDED as } r \to 0 \]
\[ \Rightarrow \quad B = 0 \]
\[ \Rightarrow \quad R(r) = Ar^n \]

\((6)\) \quad n = 1, 2, 3, \ldots \text{ ...}

CASE \quad n = 0

Here, \((5)\) becomes
\[ r^2 R'' + rR' = 0 \]
\[ \Rightarrow \quad \frac{R''}{R'} = -\frac{1}{r} \]
Integrating both sides

\[ \ln R' = -\ln r + C_1 \]

Taking **Exponentials** on both sides

\[ R' = \frac{C}{r} \quad C = \text{Arbitrary Constant.} \]

\[ R(r) = C \ln r + D \]

\( C, D \) arbitrary constants.

For \( m(r, \theta) \) to remain **bounded**

as \( r \to 0 \) \[ c = 0 \]

\[ R(r) = A_0 \quad (\text{constant}) \]
Therefore

\[
\mu_n(r, \theta) = \begin{cases} 
A_0 & n = 0 \\
A_n r^n \cos n\theta + B_n r^n \sin n\theta 
& n = 1, 2, \ldots
\end{cases}
\]

Using superposition

\[
\mu(r, \theta) = \sum_{n=0}^{\infty} A_n r^n \cos n\theta + \sum_{n=1}^{\infty} B_n r^n \sin n\theta
\]

\[
0 \leq r \leq a \\
-\pi \leq \theta \leq \pi
\]

APPLICATION OF BC \( \mu(a, \theta) = f(\theta) \)

\[
\mu(a, \theta) = \sum_{n=0}^{\infty} A_n a^n \cos n\theta + \sum_{n=1}^{\infty} B_n a^n \sin n\theta
\]

\[
= f(\theta)
\]
This is a Fourier Series for the function $f(\theta)$.

\[ A_n = \frac{1}{\pi a^n} \int_{-\pi}^{\pi} f(\theta) \cos n\theta \, d\theta \quad n = 1, 2, \ldots \]

\[ B_n = \frac{1}{\pi a^n} \int_{-\pi}^{\pi} f(\theta) \sin n\theta \, d\theta \quad n = 1, 2, \ldots \]

\[ A_0 = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(\theta) \, d\theta. \]