A multiplicity formula for tensor products of $SL_2$ modules and an explicit $Sp_{2n}$ to $Sp_{2n-2} \times Sp_2$ branching formula.

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Abstract. In the restriction of an irreducible representation of $Sp_{2n}$ to the standard $Sp_{2n-2}$ the multiplicity spaces are naturally $Sp_2 \cong SL_2$ modules. We show that these multiplicity spaces are each equivalent to a specified tensor product of $n$ irreducible $SL_2$ modules. The key to these results is a generalization of the Clebsch-Gordan formula and a result of J. Lepowsky that gives the $C_n$ branching to $C_{n-1} \times C_1$ as a difference of two simple partition functions.

1. Introduction

The purpose of this note is to give an elementary decomposition of the restriction of an irreducible representation of $C_n$ to $C_{n-1} \times C_1$. By a decomposition we mean an explicit description of the $C_1$-module structure of the multiplicity spaces that occur in the restriction of an irreducible representation of $C_n$ to $C_{n-1}$. By elementary we mean using relatively simple combinatorial methods. In principle the results of this note can be derived from those of [[4], Theorem 5.2] which uses the theory of Yangians and is far from elementary. As a byproduct of our work we derive a formula for the decomposition of arbitrary tensor products of irreducible representations of $SL_2$, generalizing the Clebsch-Gordan formula. Here the multiplicities are given as a difference of two generalized Kostant partition functions.

2. Tensor products of $SL(2, \mathbb{C})$ representations

Let $H = SL(2, \mathbb{C})$ and let $F^k$ be the irreducible representation of $H$ of dimension $k + 1$. The Clebsch-Gordan formula implies that if $r_1 \geq r_2$ then

$$F^{r_1} \otimes F^{r_2} \cong F^{r_1 + r_2} \oplus F^{r_1 + r_2 - 2} \oplus \cdots \oplus F^{r_1 - r_2}. \tag{2.1}$$

In this section we extend the Clebsch-Gordan formula to an arbitrary tensor product of representations of $H$.

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We begin by setting up some notation. Let \( \{v_1, \ldots, v_n\} \) be the standard basis for \( \mathbb{R}^n \) and set \( \Sigma_n = \{v_1 \pm v_n, \ldots, v_{n-1} \pm v_n\} \). We identify \( \mathbb{R}^n \) with \((\mathbb{R}^n)^*\); thus if \( v \in \mathbb{R}^n \), \( e^v \) is a function on \((\mathbb{R}^n)^*\). Denote by \( \mathcal{P}_n(v) \) the coefficient of \( e^v \) in the formal product
\[
\prod_{w \in \Sigma_n} \frac{1}{1 - e^w}.
\]
This says that \( \mathcal{P}_n(v) \) is the number of ways of writing
\[
v = \sum_{w \in \Sigma_n} c_w w, \quad c_w \in \mathbb{N}.
\]
Finally let
\[
m_l(r_1, \ldots, r_n) = \dim \text{Hom}_H(F^l, F^{r_1} \otimes \cdots \otimes F^{r_n}).
\]
The following is a reinterpretation of formula (2.1).

**Lemma 2.1.** Let \( r_1, r_2, l \in \mathbb{N} \). Then
\[
m_l(r_1, r_2) = \mathcal{P}_2(r_1 v_1 + r_2 v_2 - l v_2) - \mathcal{P}_2(r_1 v_1 + r_2 v_2 + (l + 2) v_2).
\]

**Proof.** Note that \( \mathcal{P}_2(aw_1 + bw_2) = 1 \) if and only if \( b \in \{-a, 2 - a, \ldots, a - 2, a\} \).
The result follows by considering the cases \( r_1 \leq r_2 \) and \( r_1 > r_2 \) separately. \( \square \)

The result of this section is a generalization of Lemma 2.1 to a tensor product of an arbitrary number of irreducible \( H \)-modules. First we develop some combinatorial properties of \( \mathcal{P}_n \).

Let \( \Sigma^+_n = \{v_1 + v_n, \ldots, v_{n-1} + v_n\} \) and \( \Sigma^-_n = \{v_1 - v_n, \ldots, v_{n-1} - v_n\} \). Denote by \( \mathcal{P}^\pm_n(v) \) the coefficient of \( e^v \) in
\[
\prod_{w \in \Sigma^\pm_n} \frac{1}{1 - e^w}.
\]
It is easy to see that
\[
\mathcal{P}_n(v) = \sum_{u+w=v} \mathcal{P}^+_n(u) \mathcal{P}^-_n(w).
\]
Since \( \Sigma^+_n, \Sigma^-_n \) are linearly independent the corresponding partition functions take only values 0 or 1. Furthermore, one can easily check that
\[
\mathcal{P}^+_n(a_1 v_1 + \cdots + a_n v_n) = 1 \iff a_1, \ldots, a_{n-1} \in \mathbb{N} \quad \text{and} \quad \sum_{j=1}^{n-1} a_j = a_n
\]
\[
\mathcal{P}^-_n(b_1 v_1 + \cdots + b_n v_n) = 1 \iff b_1, \ldots, b_{n-1} \in \mathbb{N} \quad \text{and} \quad \sum_{j=1}^{n-1} b_j = -b_n
\]
Let \( v = c_1 v_1 + \cdots + c_n v_n \) and suppose \( v = u + w \) with \( u = a_1 v_1 + \cdots + a_n v_n \) and \( w = b_1 v_1 + \cdots + b_n v_n \). Then \( a_j + b_j = c_j \) for \( j = 1, \ldots, n \). If \( \mathcal{P}^+_n(u) \mathcal{P}^-_n(w) = 1 \) then
\[
c_n = \sum_{j=1}^{n-1} a_j - b_j.
\]
(2.2)

Define a bisection of a natural number \( m \) to be a two-part partition of \( m \). Then \( \mathcal{P}_n(v) \) counts the number of bisections of \( c_1, \ldots, c_{n-1} \) that satisfy (2.2). This description provides a useful recursive formula.
**Lemma 2.2.**

\[ \mathcal{P}_n(c_1v_1 + \cdots + c_nv_n) = \sum_{i=0}^{c_n-1} \mathcal{P}_{n-1}(c_1v_1 + \cdots + c_{n-2}v_{n-2} + (c_{n-1} + c_n - 2i)v_{n-1}) \]

**Proof.** The \(i\)th summand on the right hand side counts the number of bisections of \(c_1, \ldots, c_{n-2}\) that satisfy \(c_{n-1} + c_n - 2i = \sum_{j=1}^{n-2} a_j - b_j\). (Here \(c_j = a_j + b_j\) for \(j = 1, \ldots, n - 2\).) These bisections correspond to the bisections of \(c_1, \ldots, c_{n-1}\) that satisfy \(c_n = \sum_{j=1}^{n-1} a_j - b_j\) with \(a_{n-1} = i\) and \(b_{n-1} = c_{n-1} - i\).

**Theorem 2.3.** Let \(r_1, \ldots, r_n, l \in \mathbb{N}\). Then

\[ m_l(r_1, ..., r_n) = \mathcal{P}_n(r_1v_1 + \cdots + r_nv_n - lv_n) - \mathcal{P}_n(r_1v_1 + \cdots + r_nv_n + (l + 2)v_n). \]

**Proof.** We proceed by induction on \(n \geq 2\). If \(n = 2\) use Lemma 2.1. Now suppose \(n > 2\) and the claim holds for \(n - 1\). Let \(r_1, \ldots, r_n, l \in \mathbb{N}\) and to simplify matters write \(S_k = \sum_{j=1}^{k} r_jv_j\) and \(Q(t) = \mathcal{P}_{n-1}(S_{n-2} + tv_{n-1})\). By Lemma 2.2 we obtain

\[ \mathcal{P}_n(S_n - lv_n) - \mathcal{P}_n(S_n + (l + 2)v_n) = \sum_{i=0}^{r_{n-1}} Q(r_{n-1} + r_n - 2i - l) - Q(r_{n-1} + r_n - 2i + l + 2). \]

If \(r_{n-1} \leq r_n\) then \(r_{n-1} + r_n - 2i \geq 0\) so by the inductive hypothesis

\[ Q(r_{n-1} + r_n - 2i - l) - Q(r_{n-1} + r_n - 2i + l + 2) = m_l(r_1, ..., r_{n-2}, r_{n-1} + r_n - 2i). \]

By the Clebsch-Gordan formula

\[ \sum_{i=0}^{r_{n-1}} m_l(r_1, ..., r_{n-2}, r_{n-1} + r_n - 2i) = m_l(r_1, ..., r_{n-2}, r_{n-1}, r_n). \]

If \(r_{n-1} > r_n\) the situation is not as straightforward. As above we have

\[ \mathcal{P}_n(S_n - lv_n) - \mathcal{P}_n(S_n + (l + 2)v_n) = m_l(r_1, ..., r_{n-2}, r_{n-1}, r_n) + E \]

where

\[ E = \sum_{i=r_{n-1}+1}^{r_{n-1}} Q(r_{n-1} + r_n - 2i - l) - Q(r_{n-1} + r_n - 2i + l + 2). \]

Rewrite \(E\) as

\[ \sum_{i=1}^{r_{n-1} - r_n} Q(r_{n-1} - r_n - 2i - l) - Q(r_{n-1} - r_n - 2i + l + 2) \]

and notice that

\[ r_{n-1} - r_n - 2i - l = -(r_{n-1} - r_n - 2(r_{n-1} - r_n + 1 - i) + l + 2). \]

Therefore if we set \(C_i = r_{n-1} - r_n - 2i - l\) then by rearranging terms

\[ E = \sum_{i=1}^{r_{n-1} - r_n} Q(C_i) - Q(-C_i). \]

But clearly \(Q(t) = Q(-t)\) so \(E = 0\).
3. An application to $Sp_{2n}$ branching

Label a basis for $\mathbb{C}^2l$ as $e_{\pm 1}, \ldots, e_{\pm l}$ where $e_{-i} = e_{2l+1-i}$. Here we view $\mathbb{C}^2l$ as column vectors. Denote by $s_l$ the $l \times l$ matrix with ones on the anti-diagonal and zeros everywhere else. Set

$$J_l = \begin{bmatrix} 0 & s_l \\ -s_l & 0 \end{bmatrix}$$

and define the skew-symmetric bilinear form $\Omega_l(x, y) = x^t J_l y$ on $\mathbb{C}^{2l}$. Let $G = Sp(\mathbb{C}^{2n}, \Omega_n)$ and define subgroups

$$K = \{ k \in G : ke_n = e_n \text{ and } ke_{-n} = e_{-n} \}$$

$$H = \{ h \in G : he_j = e_j \text{ for } j = \pm 1, \ldots, \pm n - 1 \}$$

Then $K \cong Sp(\mathbb{C}^{2(n-1)}, \Omega_{n-1})$ and $H \cong Sp(\mathbb{C}^2, \Omega_1) \cong SL(2, \mathbb{C})$. Let $\Lambda = (\Lambda_1 \geq \cdots \geq \Lambda_n \geq 0)$ be a decreasing sequence of natural numbers. We identify the set of such $\Lambda$ with the dominant integral weights of $G$ as in [[1], Proposition 2.5.11]. Let $V^\Lambda$ be the finite dimensional irreducible regular representation of $G$ of high weight $\Lambda$. Similarly a decreasing sequence of $n-1$ natural numbers $\mu = (\mu_1 \geq \cdots \geq \mu_{n-1} \geq 0)$ is identified with the corresponding dominant integral weights of $K$. Let $V^\mu$ be the finite dimensional irreducible regular representation of $K$ of high weight $\mu$.

We say $\mu$ doubly interlaces $\Lambda$ if $\Lambda_i \geq \mu_i \geq \Lambda_{i+2}$ for $i = 1, \ldots, n-1$ (with $\Lambda_{n+1} = 0$). Given $\Lambda, \mu$ set $r_i(\Lambda, \mu) = x_i - y_i$, where $\{x_1 \geq y_1 \geq \cdots \geq x_n \geq y_n\}$ is the decreasing rearrangement of $\{\Lambda_1, \ldots, \Lambda_n, \mu_1, \ldots, \mu_{n-1}, 0\}$.

**Theorem 3.1** ([1], Proposition 8.1.5). Let $n \geq 2$. Then $\dim Hom_K(V^\mu, V^\Lambda) > 0$ if and only if $\mu$ doubly interlaces $\Lambda$. If $\mu$ doubly interlaces $\Lambda$ then $\dim Hom_K(V^\mu, V^\Lambda) = \prod_{j=1}^n (r_i(\Lambda, \mu) + 1)$.

This theorem in particular provides the decomposition of $K$ modules

$$V^\Lambda \cong \bigoplus_{\mu} V^\mu \otimes Hom_K(V^\mu, V^\Lambda)$$

where the sum is over all $\mu$ that doubly interlaces $\Lambda$. Here $K$ acts on left factor. Since $H$ is a subgroup of the centralizer of $K$ in $G$, $H$ acts on the multiplicity spaces $Hom_K(V^\mu, V^\Lambda)$. One is thus led to the natural question: what is the $H$-module structure of $H_K(\mu, \Lambda) = Hom_K(V^\mu, V^\Lambda)$?

The following theorem, due to J. Lepowsky ([3]), provides a partial answer.

**Theorem 3.2** ([2], Proposition 9.5.9). Let $\Lambda, \mu$ be as above and set $r_i = r_i(\Lambda, \mu)$. Then

$$\dim Hom_H(F^l, H_K(\mu, \Lambda)) = \mathcal{P}_n(r_1v_1 + \cdots + r_nv_n - lv_n) - \mathcal{P}_n(r_1v_1 + \cdots + r_nv_n + (l+2)v_n).$$

We combine this result with Theorem 2.3 to obtain an explicit decomposition of $V^\Lambda$ as a $K \times H$ module.

**Theorem 3.3.** Let $\Lambda, \mu$ be as above and set $r_i = r_i(\Lambda, \mu)$. Then as a $K \times H$-module

$$V^\Lambda \cong \bigoplus_{\mu} V^\mu \otimes (F^{r_1} \otimes \cdots \otimes F^{r_n}).$$

The direct sum is over all $\mu$ that doubly interlace $\Lambda$. 
References


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