1. Find a solution to the Chinese Postman Problem in this graph, given that every edge has equal weight.

Solution.

The problem is to find the shortest closed walk which uses every edge at least once. The graph is not Eulerian and so some edges will have to be duplicated in order to find a closed walk using each edge at least once. Since there are 8 vertices of odd degree in the graph at least 4 edges will have to be duplicated, and since the 8 vertices are adjacent to one another in pairs, it is easy to see that a solution results if we add the 4 dotted edges as shown below.

Any Euler trail in this graph is a solution. One such trail is $abeghlkjiefgkonjfbcdhlpnmea$.

2. (i) Prove that a graph in which the degree of each vertex is at least two contains a cycle.

(ii) Prove that a tree with at least 2 vertices has at least 2 vertices of degree 1.

Solution.

(i) The result is trivial if the graph has loops or multiple edges, so suppose the graph is simple. Construct a walk $v_1 \to v_2 \to v_3 \ldots$ in the graph by choosing any vertex as $v_1$, any vertex adjacent to $v_1$ as $v_2$, any vertex adjacent to $v_2$ other than $v_1$ as $v_3$, and so on. That is, at each vertex $v_i$, choose $v_{i+1} \neq v_{i-1}$. The existence of such a vertex is guaranteed by the fact that the degree of each vertex is at least 2. Since the graph has only finitely many vertices, at some stage a vertex will be repeated. Then there is a cycle between the two occurrences of the repeated vertex.
(ii) Note that part (i) is equivalent to:

A graph with no cycles (and at least one edge) has at least one vertex with degree 1.

Since a tree is defined as a connected graph without cycles, it follows that a tree has at least one vertex of degree 1. Now construct a path as follows:

Find a vertex of degree 1, and move to its adjacent vertex. If that vertex has degree 1, then we have found 2 vertices of degree 1. Otherwise, its degree is at least 2, and we can choose another edge along which to move to a third vertex. Keep doing this, going along previously unused edges as long as possible.

Since there are no cycles, we will never get back to an already visited vertex, and since there are only finitely many vertices, we cannot keep getting new vertices forever. Therefore, we must eventually reach a new vertex from which we are unable to continue, i.e., a vertex not previously visited, and with no unused edge incident to it. That is, we have found a second vertex of degree 1.

Here is an alternative proof:

Suppose there are \( n > 1 \) vertices, and \( p \) of them have degree 1. Hence the other \( n - p \) vertices have degree \( > 1 \), i.e., \( \geq 2 \) (since the tree is connected, with at least two vertices, so no vertex has degree 0). Now recall that a tree with \( n \) vertices has \( n - 1 \) edges.

By the Handshaking lemma,

\[
2(\text{number of edges}) = 2(n - 1) = (\text{total degree}) 
\geq 1 \times p + 2(n - p) = 2n - p, \\
\text{which gives } p \geq 2.
\]

3 Prove, directly from the definition of a tree as a connected graph without cycles, that the addition of one edge to a tree creates exactly one cycle.

\textbf{Solution.}

Firstly, since a tree has no cycles it cannot have multiple edges or loops, and so there is at most one edge between any pair of vertices.

If an edge is added to a pair of adjacent vertices, then exactly one cycle is formed, consisting of the the one existing edge and the added edge.

Now suppose that an edge is added to a pair of non-adjacent vertices, \( v_i \) and \( v_j \). Since a tree is connected there is a path from \( v_i \) to \( v_j \), and so adding the edge \( v_i v_j \) forms the cycle \( v_i \ldots v_j v_i \). More than one cycle could be formed only if there were at least two paths from \( v_i \) to \( v_j \). But if there were more than one path between \( v_i \) and \( v_j \) in the tree, then there would be a cycle. Hence, there is exactly one path between any pair of vertices, and exactly one cycle created by adding an edge.
4. How many edges are there in a forest with \( v \) vertices and \( k \) components?

Solution.
Suppose the number of vertices in the \( i \)th component of the forest is \( v_i \). Each component of a forest is a tree, and so the number of edges in the \( i \)th component is \( v_i - 1 \). The total number of edges is therefore

\[
\sum_{i=1}^{k} (v_i - 1) = \sum_{i=1}^{k} v_i - \sum_{i=1}^{k} 1 = v - k.
\]

5. Let \( T \) be a tree with \( p \) vertices of degree 1 and \( q \) other vertices. Show that the sum of the degrees of the vertices of degree greater than 1 is \( p + 2(q - 1) \).

Solution.
The number of vertices is \( p + q \), and so the number of edges is \( e = p + q - 1 \). Hence, by the Handshaking Lemma,

\[
\text{Total degree} = \sum_{v_i} \text{deg}(v_i)
= p + \sum_{\text{deg}(v_i) > 1} \text{deg}(v_i)
= 2e
= 2(p + q - 1),
\]

and so \( \sum_{\text{deg}(v_i) > 1} \text{deg}(v_i) = 2(p + q - 1) - p = p + 2(q - 1) \).

6. Show that if a tree has two vertices of degree 3, then it must have at least 4 vertices of degree 1.

Solution.
Let \( v \) be the number of vertices, and \( p \) the number of vertices of degree 1. If two vertices have degree 3, there are \( (v - p - 2) \) vertices with degree other than 1 or 3. In particular, each of these \( (v - p - 2) \) vertices has degree at least 2.

\[
\sum \text{degrees} \geq p + 2 \times 3 + 2(v - p - 2) = 2v - p + 2.
\]

But \( \sum \text{degrees} = 2 \times \text{number of edges} = 2(v - 1) \).

Therefore \( 2v - 2 \geq 2v - p + 2 \), or \( p \geq 4 \).
7. Show that, for each value of \( n \geq 1 \), the graph associated with the alcohol molecule \( C_nH_{2n+1}OH \) is a tree. (Carbon, Hydrogen, Oxygen have valencies 4, 1, 2 respectively.)

Solution.

Since a molecule is a collection of atoms connected by chemical bonds, the associated graph is connected. It consists of \( n \) vertices of degree 4 each labelled \( C \), \( 2n+2 \) vertices of degree 1 each labelled \( H \), and 1 vertex of degree 2 labelled \( O \).

Hence the number of vertices is \( v = n + (2n + 2) + 1 = 3n + 3 \), and, using the Handshaking Lemma, the number of edges is \( e = \frac{1}{2}(4n + 1(2n + 2) + 2) = 3n + 2 \). Since \( e = v - 1 \), and the graph is connected, it must be a tree.

8. (i) Find the number of molecules with formula \( C_5H_{12} \), and draw them.
(ii) How many non-isomorphic trees are there with 5 vertices?
(iii) Comment on the relationship between your answers to parts (i) and (ii).

Solution.

(i) As in the previous question, the graph associated with any such molecule is connected, the number of vertices is 17, and the number of edges is \( \frac{4 \times 5 + 12}{2} = 16 \). So the graph is a tree, and therefore must be simple. (So, for example, there can be no double bonds like \( \cdots C == C \cdots \).)

Moreover, each \( C \) must be adjacent to at least one other \( C \), else we would have a component of the graph of the form

\[
\begin{align*}
H & \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad H \\
H & \quad C \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad H \\
& \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad H
\end{align*}
\]

disconnected from the rest of the graph (since each \( H \) has valency only 1).

Hence the \( C \)s and the edges between them must form a connected subgraph — a subtree in fact.

The 5 carbon atoms could be connected up with single bonds in three essentially different (non-isomorphic) ways:

\[
\begin{array}{ccc}
C & C & C & C & C \\
C & C & C & C & C \\
C & C & C & C & C
\end{array}
\]

In each case there is an essentially unique way to add the \( H \)s, giving the following 3 isomers of \( C_5H_{12} \) (where \( C = \bullet \), \( H = \bullet \)):
There are 3 non-isomorphic trees on 5 vertices:

(iii) The 3 trees in part (ii) are precisely those formed by the carbon atoms in part (i).

9. Repeat question 7 for $C_6H_{14}$, and trees with 6 vertices.

Solution.

(i) There are 5 isomers of $C_6H_{14}$, given by the following 5 arrangements of the carbon atoms:

Adding the $H$s in the obvious way gives the 5 molecules.

(ii) There are 6 non-isomorphic trees on 6 vertices – the 5 shown in part (i), and the following:

(iii) In this case, the 6 carbon atoms can be arranged in a way which corresponds to only 5 of the 6 non-isomorphic trees on 6 vertices. (In the sixth tree, one vertex has degree 5, and carbon has valency 4.)

10. Find the Prüfer sequence corresponding to each of the following labelled trees:

Solution.

We find the lowest labelled leaf (vertex of degree 1), delete that leaf and its incident edge, and write down the label of the vertex to which the just deleted leaf was adjacent. Continue until just two vertices are left.

Lowest leaf: 2
Remove 2, write 4

Lowest leaf: 3
Remove 3, write 4

Lowest leaf: 5
Remove 5, write 4
Lowest leaf: 4 
Remove 4, write 1 

Two left: Stop. 
Answer: (4,4,4,1)

Similarly, with the second graph: remove 1 and write 4; remove 3 and write 2; remove 5 and write 2; remove 2 and write 4; stop. The sequence is (4,2,2,4).

For the third graph the sequence is (2,2,1,3,3,1,4,4,1,5,5).

11. Find the labelled trees corresponding to these Prüfer sequences:

(i) (1, 2, 3, 4, 5)  (ii) (3, 3, 3, 3, 3)  (iii) (2, 8, 6, 3, 1, 2)

Solution.

(i) In order to find the tree corresponding to a particular Prüfer sequence 
$$(a_1, a_2, \ldots, a_{n-2})$$ 
(where each $a_i$ is one of 1, 2, \ldots, $n$), we augment the sequence to 
$$(a_1, a_2, \ldots, a_{n-2}, n)$$, and form a new sequence 
$$(b_1, b_2, \ldots, b_{n-2}, b_{n-1})$$, term by term, such that each $b_i$ is the smallest positive integer not equal to any of $b_1, b_2, \ldots, b_{i-1}, a_i, a_{i+1}, \ldots, a_{n-2}, n$. The labelled graph whose edges are $[a_1, b_1], [a_2, b_2], \ldots, [a_{n-2}, b_{n-2}], [b_{n-1}, n]$ is then the labelled tree with Prüfer sequence $(a_1, a_2, \ldots, a_{n-2})$.

From (1,2,3,4,5), (5 terms, so $n = 7$) we augment to (1,2,3,4,5,7) then form (6,1,2,3,4,5), as follows:

the least positive integer not equal to any of 1,2,3,4,5,7 is 6;
the least positive integer not equal to any of 2,3,4,5,7 or 6 is 1;
the least positive integer not equal to any of 3,4,5,7 or 6,1 is 2;
the least positive integer not equal to any of 4,5,7 or 6,1,2 is 3;
the least positive integer not equal to any of 5,7 or 6,1,2,3 is 4;
the least positive integer not equal to any of 7 or 6,1,2,3,4 is 5.

Hence the edges are [1,6], [2,1], [3,2], [4,3], [5,4], [7,5]:

\[ 6 \quad 1 \quad 2 \quad 3 \quad 4 \quad 5 \quad 7 \]

(ii) Similarly, from (3,3,3,3,3), again $n = 7$, so we augment to (3,3,3,3,3,7) then form (1,2,4,5,6,3):

edges [3,1], [3,2], [3,4], [3,5], [3,6], [7,3].

(iii) With (2,8,6,3,1,2) (6 terms), $n = 8$, so we augment to (2,8,6,3,1,2,8) then form (4,5,7,6,3,1,2):

edges [2,4], [5,8], [6,7], [3,6], [1,3], [2,1], [8,2].