Classical Lie algebras and Yangians

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Lecture 3. Yangians: representations

Recall that the Yangian $\mathrm{Y}(\mathfrak{gl}_N)$ is an associative algebra with generators $t_{ij}^{(r)}$ and the defining relations

$$(u-v)\,[t_{ij}(u), t_{kl}(v)] = t_{kj}(u)\,t_{il}(v) - t_{kj}(v)\,t_{il}(u),$$

where

$$t_{ij}(u) = \delta_{ij} + t_{ij}^{(1)} u^{-1} + t_{ij}^{(2)} u^{-2} + \dots \in \mathrm{Y}(\mathfrak{gl}_N)[[u^{-1}]].$$

Definition. A representation L of the Yangian $\mathrm{Y}(\mathfrak{gl}_N)$ is called a highest weight representation if there exists a nonzero vector $\zeta \in L$ such that L is generated by ζ and the following relations hold

$$t_{ij}(u)\,\zeta = 0 \qquad \text{for} \quad 1 \leqslant i < j \leqslant N, \qquad \text{and}$$

$$t_{ii}(u)\,\zeta = \lambda_i(u)\,\zeta \qquad \text{for} \quad 1 \leqslant i \leqslant N$$

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for some formal series

$$\lambda_i(u) = 1 + \lambda_i^{(1)} u^{-1} + \lambda_i^{(2)} u^{-2} + \dots, \qquad \lambda_i^{(r)} \in \mathbb{C}.$$

The vector ζ is called the highest vector of L, and the N-tuple of formal series $\lambda(u) = \big(\lambda_1(u), \dots, \lambda_N(u)\big)$ is the highest weight of L.

Verma module

Definition

Let $\lambda(u) = \big(\lambda_1(u), \dots, \lambda_N(u)\big)$ be an arbitrary tuple of formal series. The Verma module $M(\lambda(u))$ is the quotient of $\mathrm{Y}(\mathfrak{gl}_N)$ by the left ideal generated by all coefficients of the series $t_{ij}(u)$ for $1 \leqslant i < j \leqslant N$ and $t_{ii}(u) - \lambda_i(u)$ for $1 \leqslant i \leqslant N$.

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Proposition. For any given order on the set of generators $t_{ji}^{(r)}$ with $1 \leqslant i < j \leqslant N$ and $r \geqslant 1$, the elements

$$t_{j_1 i_1}^{(r_1)} \dots t_{j_m i_m}^{(r_m)} 1_{\lambda(u)}, \qquad m \geqslant 0,$$

with ordered products of the generators, form a basis of $M(\lambda(u))$.

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Theorem

Every finite-dimensional irreducible representation of $\mathrm{Y}(\mathfrak{gl}_N)$ is isomorphic to $L(\lambda(u))$ for some $\lambda(u)$.

Proof.

Regard the representation of $\mathrm{Y}(\mathfrak{gl}_N)$ as a $\mathfrak{gl}_N$-module using the embedding $E_{ij} \mapsto t_{ij}^{(1)}$. □

Given an N-tuple of complex numbers $\lambda = (\lambda_1, \ldots, \lambda_N)$ denote by $L(\lambda)$ the irreducible representation of the Lie algebra $\mathfrak{gl}_N$ with the highest weight λ. So, $L(\lambda)$ is generated by a nonzero vector ζ such that

$$E_{ij}\,\zeta = 0 \qquad \text{for} \quad 1 \leqslant i < j \leqslant N, \qquad \text{and}$$

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Equip $L(\lambda)$ with a structure of $\mathrm{Y}(\mathfrak{gl}_N)$-module via the evaluation homomorphism

$$t_{ij}(u) \mapsto \delta_{ij} + E_{ij}\,u^{-1}.$$

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If L and M are any two $\mathrm{Y}(\mathfrak{gl}_N)$-modules, then the tensor product space $L \otimes M$ can be equipped with a $\mathrm{Y}(\mathfrak{gl}_N)$-action with the use of the comultiplication Δ on $\mathrm{Y}(\mathfrak{gl}_N)$.

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By the coassociativity of Δ, we may unambiguously define multiple tensor product modules of the form

$$L(\lambda^{(1)}) \otimes L(\lambda^{(2)}) \otimes \dots \otimes L(\lambda^{(k)}).$$

Representations of $\mathrm{Y}(\mathfrak{gl}_2)$

Consider the irreducible highest weight representation $L(\lambda(u))$ of $\mathrm{Y}(\mathfrak{gl}_2)$ with an arbitrary highest weight $\lambda(u) = \big(\lambda_1(u), \lambda_2(u)\big)$.

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Proposition

If $\dim L(\lambda(u)) < \infty$ *then there exists a formal series*

$$f(u) = 1 + f_1 u^{-1} + f_2 u^{-2} + \dots, \qquad f_r \in \mathbb{C},$$

such that $f(u)\lambda_1(u)$ *and* $f(u)\lambda_2(u)$ *are polynomials in* u^{-1}.

let $\lambda_1(u)$ and $\lambda_2(u)$ be polynomials in u^{-1} of degree not more than k. Write the decompositions

$$\lambda_1(u) = (1 + \alpha_1 u^{-1}) \dots (1 + \alpha_k u^{-1}),$$

$$\lambda_2(u) = (1 + \beta_1 u^{-1}) \dots (1 + \beta_k u^{-1}).$$

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Proposition

Suppose that for every $i = 1, \dots, k-1$ the following condition holds: if the multiset $\{\alpha_p - \beta_q \mid i \leqslant p, q \leqslant k\}$ contains nonnegative integers, then $\alpha_i - \beta_i$ is minimal amongst them. Then the representation $L(\lambda_1(u), \lambda_2(u))$ of $\mathrm{Y}(\mathfrak{gl}_2)$ is isomorphic to the tensor product module

$$L(\alpha_1, \beta_1) \otimes L(\alpha_2, \beta_2) \otimes \dots \otimes L(\alpha_k, \beta_k).$$

Theorem

The irreducible highest weight representation $L(\lambda_1(u), \lambda_2(u))$ *of* $\mathrm{Y}(\mathfrak{gl}_2)$ *is finite-dimensional if and only if there exists a monic polynomial* $P(u)$ *in* u *such that*

$$\frac{\lambda_1(u)}{\lambda_2(u)} = \frac{P(u+1)}{P(u)}.$$

In this case $P(u)$ *is unique.*

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In this case $P(u)$ *is unique.*

The polynomial $P(u)$ is called the Drinfeld polynomial of the finite-dimensional representation $L(\lambda_1(u), \lambda_2(u))$.

Proof.

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The highest weight of the $\mathrm{Y}(\mathfrak{gl}_2)$-evaluation module is

$$\lambda_1(u) = 1 + \alpha\, u^{-1}, \qquad \lambda_2(u) = 1 + \beta\, u^{-1}.$$

Hence, if $\alpha - \beta \in \mathbb{Z}_+$ then

$$\frac{\lambda_1(u)}{\lambda_2(u)} = \frac{u + \alpha}{u + \beta} = \frac{P(u+1)}{P(u)}$$

for

$$P(u) = (u + \beta)(u + \beta + 1) \dots (u + \alpha - 1).$$

□

Recall that the Yangian $\mathrm{Y}(\mathfrak{sl}_2)$ is the subalgebra of $\mathrm{Y}(\mathfrak{gl}_2)$ which consists of the elements stable under all automorphisms of the form $T(u) \mapsto f(u)\, T(u)$.

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Corollary

The isomorphism classes of finite-dimensional irreducible representations of the Yangian $\mathrm{Y}(\mathfrak{sl}_2)$ are parameterized by monic polynomials in u. Every such representation is isomorphic to the restriction of a $\mathrm{Y}(\mathfrak{gl}_2)$-module of the form

$$L(\alpha_1, \beta_1) \otimes L(\alpha_2, \beta_2) \otimes \ldots \otimes L(\alpha_k, \beta_k),$$

where each difference $\alpha_i - \beta_i$ is a positive integer.

Irreducibility criterion

Define the string corresponding to a pair of complex numbers (α, β) with $\alpha - \beta \in \mathbb{Z}_+$ as the set

$$S(\alpha, \beta) = \{\beta, \beta + 1, \ldots, \alpha - 1\}.$$

If $\alpha = \beta$ then the set $S(\alpha, \beta)$ is regarded to be empty.

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Definition

Two strings S_1 and S_2 are in general position if either

(i) $S_1 \cup S_2$ is not a string, or

(ii) $S_1 \subset S_2$, or $S_2 \subset S_1$. □

Suppose that all differences $\alpha_i - \beta_i$ are nonnegative integers.

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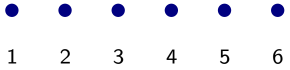
Corollary

The representation

$$L(\alpha_1, \beta_1) \otimes L(\alpha_2, \beta_2) \otimes \ldots \otimes L(\alpha_k, \beta_k)$$

of $\mathrm{Y}(\mathfrak{gl}_2)$ (*or* $\mathrm{Y}(\mathfrak{sl}_2)$) *is irreducible if and only if the strings* $S(\alpha_1, \beta_1), \ldots, S(\alpha_k, \beta_k)$ *are pairwise in general position.*

Example. The representation $L(7,1) \otimes L(6,4)$ of $\mathrm{Y}(\mathfrak{gl}_2)$ is irreducible:



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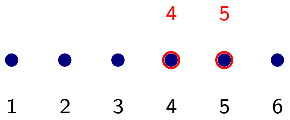

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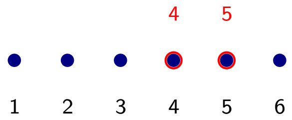

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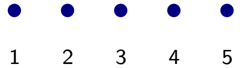

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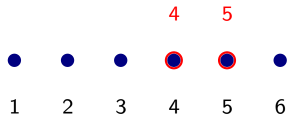

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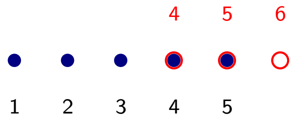

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Theorem

The irreducible highest weight representation $L(\lambda(u))$ of the Yangian $\mathrm{Y}(\mathfrak{gl}_N)$ is finite-dimensional, if and only if there exist monic polynomials $P_1(u), \dots, P_{N-1}(u)$ in u such that

$$\frac{\lambda_i(u)}{\lambda_{i+1}(u)} = \frac{P_i(u+1)}{P_i(u)}, \qquad i = 1, \dots, N-1.$$

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The polynomials $P_i(u)$ with $i = 1, \ldots, N-1$ are called the Drinfeld polynomials of $L(\lambda(u))$.

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Lemma. Suppose that L and M are finite-dimensional irreducible representations of $\mathrm{Y}(\mathfrak{gl}_N)$ with the respective sets of Drinfeld polynomials

$$\big(P_1(u), \dots, P_{N-1}(u)\big) \qquad \text{and} \qquad \big(Q_1(u), \dots, Q_{N-1}(u)\big).$$

Then the irreducible quotient of the cyclic $\mathrm{Y}(\mathfrak{gl}_N)$-span of the tensor product of the highest vectors of L and M corresponds to

$$\big(P_1(u)\, Q_1(u), \dots, P_{N-1}(u)\, Q_{N-1}(u)\big).$$

The evaluation $\mathrm{Y}(\mathfrak{gl}_N)$-module $L(\alpha+1,\dots,\alpha+1,\alpha,\dots,\alpha)$ with i copies of $\alpha+1$ is a fundamental representation; its Drinfeld polynomials are given by

$$P_i(u) = u + \alpha \qquad \text{and} \qquad P_j(u) = 1 \quad \text{if} \quad j \neq i.$$

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Corollary

Every finite-dimensional irreducible representation of the Yangian $\mathrm{Y}(\mathfrak{gl}_N)$ *is isomorphic to a subquotient of a tensor product of fundamental representations.* □

Remark

Contrary to the case $N = 2$, it is not true for $N \geqslant 3$ that every finite-dimensional irreducible representation of $\mathrm{Y}(\mathfrak{sl}_N)$ is isomorphic to a tensor product of evaluation modules. For example, the $\mathrm{Y}(\mathfrak{sl}_3)$-module $L(\lambda(u))$ with

$$\lambda_1(u) = (1 + 3\,u^{-1})(1 + u^{-1}),$$
$$\lambda_2(u) = 1 + 3\,u^{-1}, \qquad \lambda_3(u) = 1 + 2\,u^{-1}$$

is 8-dimensional. On the other hand, the possible dimensions of the evaluation modules are $1, 3, 6, 8, \dots$ so that $L(\lambda(u))$ cannot be isomorphic to a tensor product of such modules. □

Irreducibility criterion for tensor products of evaluation modules

Let the $\lambda^{(i)}$ be $\mathfrak{gl}_N$-highest weights.

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Theorem (Binary property). The $\mathrm{Y}(\mathfrak{gl}_N)$-module

$$L(\lambda^{(1)}) \otimes L(\lambda^{(2)}) \otimes \ldots \otimes L(\lambda^{(l)})$$

is irreducible if and only if the modules $L(\lambda^{(i)}) \otimes L(\lambda^{(j)})$ are irreducible for all $1 \leqslant i < j \leqslant l$.

Let

$$\lambda = (\lambda_1, \dots, \lambda_N), \qquad \mu = (\mu_1, \dots, \mu_N)$$

with $\lambda_i, \mu_i \in \mathbb{Z}$ and

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$$\lambda_1 \geqslant \cdots \geqslant \lambda_N, \qquad \mu_1 \geqslant \cdots \geqslant \mu_N.$$

We will call two disjoint finite subsets A and B of $\mathbb{Z}$ crossing if there exist elements $a_1, a_2 \in A$ and $b_1, b_2 \in B$ such that

$$a_1 < b_1 < a_2 < b_2 \qquad \text{or} \qquad b_1 < a_1 < b_2 < a_2.$$

Otherwise, A and B are called non-crossing.

For any $\mathfrak{gl}_N$-highest weight λ with integer components introduce the subset $\mathcal{A}_\lambda \subset \mathbb{Z}$ by

$$\mathcal{A}_\lambda = \{\lambda_1, \lambda_2 - 1, \dots, \lambda_N - N + 1\}.$$

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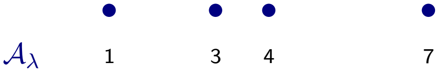
Theorem

The $\mathrm{Y}(\mathfrak{gl}_N)$-module $L(\lambda) \otimes L(\mu)$ is irreducible if and only if the sets $\mathcal{A}_\lambda \setminus \mathcal{A}_\mu$ and $\mathcal{A}_\mu \setminus \mathcal{A}_\lambda$ are non-crossing.

Example. The $\mathrm{Y}(\mathfrak{gl}_4)$-module $L(7,5,5,4) \otimes L(9,8,8,6)$ is irreducible:



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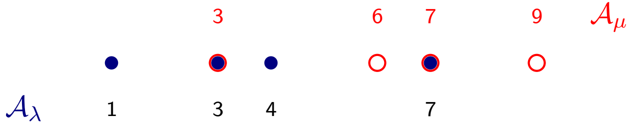

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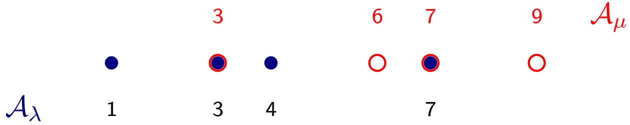

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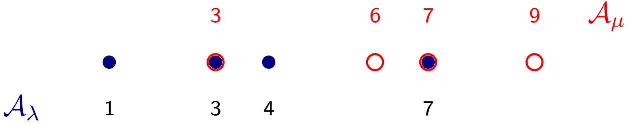

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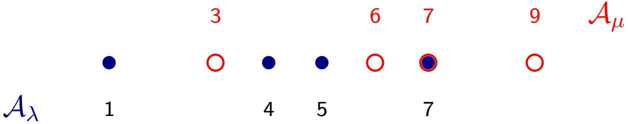

Fusion procedure

The irreducible representations of $\mathfrak{S}_k$ over $\mathbb{C}$ are parameterized by partitions of k. Given a partition λ of k denote the corresponding irreducible representation of $\mathfrak{S}_k$ by V_λ. The vector space V_λ is equipped with an $\mathfrak{S}_k$-invariant inner product $(\ ,\ )$. The orthonormal Young basis $\{v_{\mathcal{U}}\}$ of V_λ is parameterized by the set of standard λ-tableaux $\mathcal{U}$.

Set $s_i = (i, i+1)$ for $i \in \{1, \ldots, k-1\}$. We have

$$s_i \cdot v_{\mathcal{U}} = d\, v_{\mathcal{U}} + \sqrt{1-d^2}\, v_{s_i\mathcal{U}},$$

where $d = (c_{i+1} - c_i)^{-1}$ and $c_i = c_i(\mathcal{U})$ the content of the cell occupied by the number i in a standard λ-tableau $\mathcal{U}$. The tableau $s_i\mathcal{U}$ is obtained from $\mathcal{U}$ by swapping the entries i and $i+1$.

The group algebra $\mathbb{C}[\mathfrak{S}_k]$ is isomorphic to the direct sum of matrix algebras

$$\mathbb{C}[\mathfrak{S}_k] \cong \bigoplus_{\lambda \vdash k} \mathrm{Mat}_{f_\lambda}(\mathbb{C}),$$

where $f_\lambda = \dim V_\lambda$. The matrix units $e_{\mathcal{U}\mathcal{U}'} \in \mathrm{Mat}_{f_\lambda}(\mathbb{C})$ are parameterized by pairs of standard λ-tableaux $\mathcal{U}$ and $\mathcal{U}'$.

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Identify $\mathbb{C}[\mathfrak{S}_k]$ with the direct sum of matrix algebras by

$$e_{\mathcal{U}\mathcal{U}'} = \frac{f_\lambda}{k!}\,\phi_{\mathcal{U}\mathcal{U}'},$$

where $\phi_{\mathcal{U}\mathcal{U}'}$ is the matrix element corresponding to the basis vectors $v_{\mathcal{U}}$ and $v_{\mathcal{U}'}$ of the representation V_λ,

$$\phi_{\mathcal{U}\mathcal{U}'} = \sum_{s \in \mathfrak{S}_k} (s \cdot v_{\mathcal{U}}, v_{\mathcal{U}'}) \cdot s^{-1} \in \mathbb{C}[\mathfrak{S}_k].$$

For the diagonal elements we will simply write $e_{\mathcal{U}} = e_{\mathcal{U}\mathcal{U}}$ and $\phi_{\mathcal{U}} = \phi_{\mathcal{U}\mathcal{U}}$.

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The Jucys–Murphy elements of $\mathbb{C}[\mathfrak{S}_k]$ are defined by

$$x_1 = 0, \qquad x_i = (1\,i) + (2\,i) + \cdots + (i-1\,i), \quad i = 2, \ldots, k.$$

They generate a commutative subalgebra of $\mathbb{C}[\mathfrak{S}_k]$. Moreover, x_k commutes with all elements of $\mathfrak{S}_{k-1}$.

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The vectors of the Young basis are eigenvectors for the action of x_i on V_λ. For any standard λ-tableau $\mathcal{U}$ we have

$$x_i \cdot v_{\mathcal{U}} = c_i(\mathcal{U})\, v_{\mathcal{U}}, \qquad i = 1, \dots, k.$$

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Proposition (Murphy's formula). We have the relation in $\mathbb{C}[\mathfrak{S}_k]$,

$$e_{\mathcal{U}} = e_{\mathcal{V}} \frac{(x_k - a_1) \dots (x_k - a_l)}{(c - a_1) \dots (c - a_l)},$$

where $a_1, \dots, a_l$ are the contents of all addable cells of μ except for α, while c is the content of the latter.

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Equivalently,

$$e_{\mathcal{U}} = e_{\mathcal{V}} \, \frac{u - c}{u - x_k} \Big|_{u=c}.$$

For any distinct indices $i, j \in \{1, \dots, k\}$ introduce the rational function in two variables u, v with values in the group algebra $\mathbb{C}[\mathfrak{S}_k]$ by

$$\rho_{ij}(u, v) = 1 - \frac{(i\,j)}{u - v}.$$

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Proposition

Let r be a fixed index, $r \geqslant k + 1$. We have the equalities of rational functions in u valued in $\mathbb{C}[\mathfrak{S}_r]$,

$$\begin{aligned} &\phi_{\mathcal{U}}\, \rho_{k,r}(-c_k, u) \dots \rho_{1r}(-c_1, u) \\ &\qquad = \rho_{1r}(-c_1, u) \dots \rho_{k,r}(-c_k, u)\, \phi_{\mathcal{U}} \\ &\qquad\qquad = \phi_{\mathcal{U}} \Big(1 + \frac{(1\, r) + (2\, r) + \dots + (k\, r)}{u}\Big). \end{aligned}$$

Take k complex variables $u_1, \ldots, u_k$ and set

$$\phi(u_1, \ldots, u_k) = \rho_{12}(u_1, u_2)\, \rho_{13}(u_1, u_3)\, \rho_{23}(u_2, u_3)$$

$$\times \ldots\; \rho_{1k}(u_1, u_k)\, \rho_{2k}(u_2, u_k) \ldots \rho_{k-1,k}(u_{k-1}, u_k).$$

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Suppose that λ is a partition of k and let $\mathcal{U}$ be a standard λ-tableau. Set $c_i = c_i(\mathcal{U})$ for $i = 1, \ldots, k$.

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Then the consecutive evaluations

$$\phi(u_1, \dots, u_k)\big|_{u_1=c_1}\big|_{u_2=c_2} \dots \big|_{u_k=c_k}$$

of the rational function $\phi(u_1, \dots, u_k)$ are well-defined. The corresponding value coincides with the matrix element $\phi_{\mathcal{U}}$.

Example: $\lambda = (k)$. Then

$$\mathcal{U} = \quad \boxed{1 \mid 2 \mid \quad \cdots \mid k} \qquad c_i = i - 1,$$

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is the symmetrizer in $\mathbb{C}[\mathfrak{S}_k]$. By the theorem,

$$\phi_{\mathcal{U}} = \left(1 + \frac{(1\,2)}{1}\right)\left(1 + \frac{(1\,3)}{2}\right)\left(1 + \frac{(2\,3)}{1}\right)$$

$$\times \ldots \left(1 + \frac{(1\,k)}{k-1}\right)\left(1 + \frac{(2\,k)}{k-2}\right)\ldots\left(1 + \frac{(k-1\,k)}{1}\right).$$

Example: $\lambda = (1^k)$. Then

$$\mathcal{U} = \begin{array}{|c|} \hline 1 \\ \hline 2 \\ \hline \vdots \\ \hline k \\ \hline \end{array} \qquad c_i = -i + 1,$$

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$$\phi_{\mathcal{U}} = \left(1 - \frac{(1\,2)}{1}\right)\left(1 - \frac{(1\,3)}{2}\right)\left(1 - \frac{(2\,3)}{1}\right)$$

$$\times \dots \left(1 - \frac{(1\,k)}{k-1}\right)\left(1 - \frac{(2\,k)}{k-2}\right) \dots \left(1 - \frac{(k-1\,k)}{1}\right).$$

Example: $\lambda = (2, 1)$,

$$\mathcal{U} = \begin{array}{|c|c|} \hline 1 & 2 \\ \hline 3 \\ \cline{1-1} \end{array} \qquad \mathcal{V} = \begin{array}{|c|c|} \hline 1 & 3 \\ \hline 2 \\ \cline{1-1} \end{array}$$

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Then $c_1 = 0$, $c_2 = 1$, $c_3 = -1$ for $\mathcal{U}$, and

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while $c_1 = 0$, $c_2 = -1$, $c_3 = 1$ for $\mathcal{V}$, and

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Example: $\lambda = (2^2)$,

$$\phi(u_1, u_2, u_3, u_4) = \rho_{12}(u_1, u_2)\,\rho_{13}(u_1, u_3)\,\rho_{23}(u_2, u_3) \times \rho_{14}(u_1, u_4)\,\rho_{24}(u_2, u_4)\,\rho_{34}(u_3, u_4).$$

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Take the standard λ-tableau

$$\mathcal{U} = \begin{array}{|c|c|}\hline 1 & 2 \\ \hline 3 & 4 \\ \hline \end{array}$$

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Take the standard λ-tableau

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The contents are $c_1 = 0, \quad c_2 = 1, \quad c_3 = -1, \quad c_4 = 0.$

Taking $u_1 = 0, \quad u_2 = 1, \quad u_3 = -1, \quad u_4 = u$ we get

$$\phi(0,1,-1,u) = \Big(1 + (1\,2)\Big)\Big(1 - (1\,3)\Big)\Big(1 - \frac{(2\,3)}{2}\Big)$$

$$\times \Big(1 + \frac{(1\,4)}{u}\Big)\Big(1 + \frac{(2\,4)}{u-1}\Big)\Big(1 + \frac{(3\,4)}{u+1}\Big).$$

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$$\times \Big(1 + \frac{(1\,4)}{u}\Big)\Big(1 + \frac{(2\,4)}{u-1}\Big)\Big(1 + \frac{(3\,4)}{u+1}\Big).$$

By the theorem, this rational function is regular at $u = 0$ and the corresponding value coincides with $\phi_{\mathcal{U}}$.

We have

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where

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Next step:

$$\phi_{\mathcal{V}} \left(1 + \frac{(1\,4)}{u}\right) \left(1 + \frac{(2\,4)}{u-1}\right) \left(1 + \frac{(3\,4)}{u+1}\right)$$

$$= \prod_{i=1}^{3} \left(1 - \frac{1}{(u-c_i)^2}\right) \frac{u}{u-c_4} \cdot \phi_{\mathcal{V}} \frac{u-c_4}{u-x_4},$$

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where $\quad c_1=0,\quad c_2=1,\quad c_3=-1,\quad c_4=0\quad$ and

$$x_4=(1\,4)+(2\,4)+(3\,4).$$

Finally, apply Murphy's formula to get

$$\prod_{i=1}^{3}\left(1-\frac{1}{(u-c_i)^2}\right)\frac{u}{u-c_4}\cdot\phi_{\mathcal{V}}\,\frac{u-c_4}{u-x_4}\Big|_{u=c_4}=\phi_{\mathcal{U}}.$$

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Thus,

$$\phi_{\mathcal{U}}=\phi(0,1,-1,0)$$

$$=\frac{1}{2}\Big(1+(1\,2)\Big)\Big(1-(1\,3)\Big)\Big(2-(2\,3)\Big)$$

$$\times\Big(2-(1\,4)-(2\,4)-(3\,4)\Big)\Big(2+(1\,4)+(2\,4)+(3\,4)\Big).$$

The symmetric group $\mathfrak{S}_k$ acts naturally on the tensor product space

$$\mathbb{C}^N \otimes \mathbb{C}^N \otimes \ldots \otimes \mathbb{C}^N, \qquad k \text{ factors},$$

by permuting the factors. On the other hand, $\mathbb{C}^N$ carries the vector representation of the Lie algebra $\mathfrak{gl}_N$ so that the tensor product space is a representation of $\mathfrak{gl}_N$.

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by permuting the factors. On the other hand, $\mathbb{C}^N$ carries the vector representation of the Lie algebra $\mathfrak{gl}_N$ so that the tensor product space is a representation of $\mathfrak{gl}_N$.

Suppose that $\lambda = (\lambda_1, \ldots, \lambda_N)$ is a partition of k with $\ell(\lambda) \leqslant N$. Consider an arbitrary standard λ-tableau $\mathcal{U}$ and let $\Phi_{\mathcal{U}} \in \mathrm{End}\,(\mathbb{C}^N)^{\otimes k}$ denote the image of the matrix element $\phi_{\mathcal{U}}$ under the action of $\mathfrak{S}_k$ on the tensor product space.

Then the subspace

$$L_{\mathcal{U}} = \Phi_{\mathcal{U}}(\mathbb{C}^N)^{\otimes k}$$

is a $\mathfrak{gl}_N$-submodule of the tensor product module. This submodule is irreducible and isomorphic to $L(\lambda)$.

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If $\mathcal{U} = \mathcal{U}^r$ is the row tableau of shape λ, then the subspace $L_{\mathcal{U}^r}$ coincides with the image of the Young symmetrizer,

$$L_{\mathcal{U}^r} = H_{\mathcal{U}^r} A_{\mathcal{U}^r}(\mathbb{C}^N)^{\otimes k},$$

where $H_{\mathcal{U}^r}$ and $A_{\mathcal{U}^r}$ are the row symmetrizer and column anti-symmetrizer of $\mathcal{U}^r$.

In the vector representation $\mathbb{C}^N$ of $\mathfrak{gl}_N$ we have $E_{ij} \mapsto e_{ij}$ and so the image of the matrix $E = \sum_{i,j=1}^N e_{ij} \otimes E_{ij}$ under the action of $\mathfrak{gl}_N$ can be written as

$$\sum_{a=1}^{k} \sum_{i,j=1}^{N} e_{ij} \otimes 1^{\otimes(a-1)} \otimes e_{ij} \otimes 1^{\otimes(k-a)} \in \mathrm{End}\, \mathbb{C}^N \otimes \mathrm{End}\, (\mathbb{C}^N)^{\otimes k}.$$

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Hence, under the evaluation homomorphism

$$T(u) \mapsto 1 + E\, u^{-1},$$

the image of $T^{\,t}(u)$ in the representation $L_{\mathcal{U}}$ is

$$T^{\,t}(u) \mapsto 1 + \big(P_{01} + P_{02} + \cdots + P_{0k}\big)\, u^{-1}.$$

In particular, if $k = 1$ then this takes the form

$$T^{t}(u) \mapsto R_{01}(-u),$$

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For any complex number z we can make the vector space $\mathbb{C}^N$ into a representation of $\mathrm{Y}(\mathfrak{gl}_N)$ by the assignment

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More generally, $\mathrm{Y}(\mathfrak{gl}_N)$ acts on $(\mathbb{C}^N)^{\otimes k}$ by

$$T^t(u) \mapsto R_{01}(-u-z_1)\, R_{02}(-u-z_2) \dots R_{0k}(-u-z_k),$$

where $z_1, \dots, z_k$ are fixed complex numbers.

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Proposition

The subspace $L_{\mathcal{U}}$ of $(\mathbb{C}^N)^{\otimes k}$ is stable under the action of $\mathrm{Y}(\mathfrak{gl}_N)$ defined by

$$T^{\,t}(u) \mapsto R_{01}(-u - c_1)\, R_{02}(-u - c_2) \ldots R_{0k}(-u - c_k).$$

Moreover, the representation of $\mathrm{Y}(\mathfrak{gl}_N)$ on $L_{\mathcal{U}}$ obtained by restriction is isomorphic to the evaluation module $L(\lambda)$.

Proof.

Observe that $R_{ij}(u - v)$ coincides with the image of the element $\rho_{ij}(u, v)$ under the action of the symmetric group $\mathfrak{S}_{k+1}$ on the tensor product of the vector spaces $\mathbb{C}^N$. Hence, applying the fusion procedure, we get

$$R_{01}(-u - c_1)\, R_{02}(-u - c_2) \dots R_{0k}(-u - c_k)\, \Phi_{\mathcal{U}} = \Phi_{\mathcal{U}}\Big(1 + \frac{P_{01} + P_{02} + \dots + P_{0k}}{u}\Big).$$

This implies the first part of the proposition. The second part follows by taking into account that $P_{01} + P_{02} + \dots + P_{0k}$ commutes with $\Phi_{\mathcal{U}}$. □

Gelfand–Tsetlin bases

Given any finite-dimensional irreducible representation of the Yangian $\mathrm{Y}(\mathfrak{gl}_N)$, there exists an automorphism of $\mathrm{Y}(\mathfrak{gl}_N)$ of the form $T(u) \mapsto f(u)\, T(u)$ such that its composition with the representation is isomorphic to a subquotient of a tensor product module

$$L(\lambda^{(1)}) \otimes \ldots \otimes L(\lambda^{(p)}),$$

where $L(\lambda^{(i)})$ is the irreducible representation of $\mathfrak{gl}_N$ with the highest weight $\lambda^{(i)}$.

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All generators $t_{ij}^{(r)}$ with $r \geqslant p+1$ act as zero operators.

Definition

For any positive integer p, the Yangian of level p is the quotient $\mathrm{Y}_p(\mathfrak{gl}_N)$ of the algebra $\mathrm{Y}(\mathfrak{gl}_N)$ by the ideal generated by all elements $t_{ij}^{(r)}$ with $r \geqslant p+1$ and $1 \leqslant i, j \leqslant N$.

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The composition of any finite-dimensional irreducible representation of $\mathrm{Y}(\mathfrak{gl}_N)$ with an appropriate automorphism $T(u) \mapsto f(u)\, T(u)$ can be regarded as a representation of $\mathrm{Y}_p(\mathfrak{gl}_N)$ for some $p \geqslant 1$. If $p = 1$ then the algebra $\mathrm{Y}_1(\mathfrak{gl}_N)$ is isomorphic to the universal enveloping algebra $\mathrm{U}(\mathfrak{gl}_N)$.

$\mathrm{Y}_p(\mathfrak{gl}_N)$ can be regarded as an algebra with generators $t_{ij}^{(r)}$ for $1 \leqslant r \leqslant p$ and $1 \leqslant i, j \leqslant N$, subject to the defining relations

$$(u-v)\,[\,T_{ij}(u), T_{kl}(v)] = T_{kj}(u)\,T_{il}(v) - T_{kj}(v)\,T_{il}(u),$$

where

$$T_{ij}(u) = \delta_{ij}\,u^p + t_{ij}^{(1)}\,u^{p-1} + \dots + t_{ij}^{(p)}.$$

The irreducible representation $L(\lambda(u))$ is generated by a nonzero vector ζ such that

$$T_{ij}(u)\,\zeta = 0 \qquad \text{for} \quad 1 \leqslant i < j \leqslant N, \qquad \text{and}$$

$$T_{ii}(u)\,\zeta = \lambda_i(u)\,\zeta \qquad \text{for} \quad 1 \leqslant i \leqslant N,$$

where $\lambda_i(u)$ is a monic polynomial in u of degree p. Write

$$\lambda_i(u) = (u + \lambda_i^{(1)})(u + \lambda_i^{(2)}) \dots (u + \lambda_i^{(p)}), \qquad i = 1, \dots, N.$$

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Impose the generality condition

$$\lambda_i^{(k)} - \lambda_j^{(m)} \notin \mathbb{Z}, \qquad \text{for all } \ i, j \quad \text{and all } \ k \neq m.$$

The Gelfand–Tsetlin pattern $\Lambda(u)$ (associated with the highest weight $\lambda(u)$) is an array of monic polynomials in u of degree p of the form

$$\begin{array}{ccccccc}
\lambda_{N1}(u) & & \lambda_{N2}(u) & & \cdots & & \lambda_{NN}(u) \\
& \lambda_{N-1,1}(u) & & \cdots & & \lambda_{N-1,N-1}(u) & \\
& & \cdots & & \cdots & & \\
& & \lambda_{21}(u) & & \lambda_{22}(u) & & \\
& & & \lambda_{11}(u) & & &
\end{array}$$

Here the top row coincides with $\lambda(u)$, and we have the

betweenness conditions

$$\lambda_{r+1,i}(u) \longrightarrow \lambda_{ri}(u) \longrightarrow \lambda_{r+1,i+1}(u)$$

$$\text{for} \quad r = 1, \dots, N-1 \quad \text{and} \quad i = 1, \dots, r.$$

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Notation

$$\lambda_i(u) \longrightarrow \mu_i(u)$$

means that there exists a uniquely determined decomposition

$$\mu_i(u) = (u + \mu_i^{(1)})(u + \mu_i^{(2)}) \dots (u + \mu_i^{(p)}), \qquad i = 1, \dots, N-1,$$

such that $\lambda_i^{(k)} - \mu_i^{(k)} \in \mathbb{Z}_+$ for all i and k.

Theorem

The representation $L(\lambda(u))$ *of* $\mathrm{Y}_p(\mathfrak{gl}_N)$ *admits a basis* $\{\zeta_\Lambda\}$ *parameterized by all patterns* $\Lambda(u)$ *associated with the highest weight* $\lambda(u)$.

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Corollary (Branching rule).

$$L(\lambda(u))|_{\mathrm{Y}_p(\mathfrak{gl}_{N-1})} \cong \bigoplus_{\mu(u)} L'(\mu(u)),$$

where $\mu(u)$ runs over all tuples of monic polynomials $\mu(u) = \big(\mu_1(u), \dots, \mu_{N-1}(u)\big)$ of degree p satisfying the betweenness conditions.

Introduce the polynomials with coefficients in $\mathrm{Y}_p(\mathfrak{gl}_N)$ by

$$A_r(u) = T^{\,1\dots\, r}_{\,1\dots\, r}(u), \qquad B_r(u) = T^{\,1\dots\, r}_{\,1\dots\, r-1,\, r+1}(u),$$

$$C_r(u) = T^{\,1\dots\, r-1,\, r+1}_{\,1\dots\, r}(u).$$

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$$A_r(u) = T^{\,1\dots\, r}_{\,1\dots\, r}(u), \qquad B_r(u) = T^{\,1\dots\, r}_{\,1\dots\, r-1,\, r+1}(u),$$

$$C_r(u) = T^{\,1\dots\, r-1,\, r+1}_{\,1\dots\, r}(u).$$

The coefficients of the polynomials $A_r(u)$ for $r = 1, \dots, N$ and the polynomials $B_r(u)$ and $C_r(u)$ for $r = 1, \dots, N-1$ generate the algebra $\mathrm{Y}_p(\mathfrak{gl}_N)$.

For a pattern $\Lambda(u)$ due to the generality condition there exist uniquely determined decompositions

$$\lambda_{ri}(u) = (u + \lambda_{ri}^{(1)}) \dots (u + \lambda_{ri}^{(p)}), \qquad 1 \leqslant i \leqslant r \leqslant N,$$

such that $\lambda_{Ni}^{(k)} = \lambda_i^{(k)}$,

$$\lambda_{r+1,i}^{(k)} - \lambda_{ri}^{(k)} \in \mathbb{Z}_+ \qquad \text{and} \qquad \lambda_{ri}^{(k)} - \lambda_{r+1,i+1}^{(k)} \in \mathbb{Z}_+$$

for $k = 1, \dots, p$ and $1 \leqslant i \leqslant r \leqslant N - 1$.

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Set

$$l_{ri}^{(k)} = \lambda_{ri}^{(k)} - i + 1, \qquad k = 1, \dots, p \quad \text{and} \quad i = 1, \dots, r.$$

Theorem

We have

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$$B_r(-l_{ri}^{(k)})\,\zeta_\Lambda = -\lambda_{r+1,1}(-l_{ri}^{(k)})\dots\lambda_{r+1,r+1}(-l_{ri}^{(k)}-r)\,\zeta_{\Lambda+\delta_{ri}^{(k)}},$$

$$C_r(-l_{ri}^{(k)})\,\zeta_\Lambda = \lambda_{r-1,1}(-l_{ri}^{(k)})\dots\lambda_{r-1,r-1}(-l_{ri}^{(k)}-r+2)\,\zeta_{\Lambda-\delta_{ri}^{(k)}},$$

for $r = 1,\dots,N-1$.

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