Equivalences between Yangian presentations

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Joint work with Naihuan Jing and Ming Liu

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- $\big[X, J(Y)\big] = J\big([X, Y]\big)$, $\qquad J(X)$ is linear in X,
- If $\mathfrak{g} = \mathfrak{sl}_2 = \langle e, f, h\rangle$ then

  $\big[[J(e), J(f)], J(h)\big] = \big(J(e)f - eJ(f)\big)\,h$.

If $\mathfrak{g} \neq \mathfrak{sl}_2$ then consider a root space decomposition

$$\mathfrak{g} = \mathfrak{h} \oplus \bigoplus_{\alpha \in \Phi} \mathfrak{g}_\alpha,$$

where $\mathfrak{h}$ is a Cartan subalgebra, Φ is the root system.

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Choose positive roots, $\Phi = \Phi^+ \cup (-\Phi^+)$ and for each $\alpha \in \Phi^+$

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for all $h, h' \in \mathfrak{h}$.   [Guay–Nakajima–Wendlandt, 2017].

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the antipode S is an anti-automorphism of $\mathrm{Y}(\mathfrak{g})$,

$$S(X) = -X, \qquad S\big(J(X)\big) = -J(X) + \frac{1}{4}\,c_{\mathfrak{g}} X,$$

$c_{\mathfrak{g}}$ is the eigenvalue of $\omega = \sum_{k=1}^{d} X_k^2$ in the adjoint module.

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Theorem [Drinfeld, 1985]. There exists a unique series

$$\mathcal{R}(u) = 1 + \sum_{k=1}^{\infty} \mathcal{R}_k u^{-k}, \qquad \mathcal{R}_k \in \mathrm{Y}(\mathfrak{g}) \otimes \mathrm{Y}(\mathfrak{g}),$$

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$$\tau_{0,u}\Delta^{\mathrm{op}}(Y) = \mathcal{R}(u)^{-1}\big(\tau_{0,u}\Delta(Y)\big)\mathcal{R}(u) \qquad \text{for all } Y \in \mathrm{Y}(\mathfrak{g}).$$

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$$(\rho \otimes \rho)\Big(\tau_{u,v}\Delta\big(J(X)\big)\Big)R(u-v) = R(u-v)(\rho \otimes \rho)\Big(\tau_{u,v}\Delta^{\mathrm{op}}\big(J(X)\big)\Big),$$

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for all $X\in\mathfrak{g}$, up to a factor from $\mathbb{C}[[u^{-1}]]$. The factor can be chosen to make $R(u)$ a rational function in u.

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is the permutation operator

$$P : \mathbb{C}^N \otimes \mathbb{C}^N \to \mathbb{C}^N \otimes \mathbb{C}^N.$$

Example. Let $\mathfrak{g} = \mathfrak{g}_N$ which will denote

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Solving the equation, we get the R-matrix

$$R(u) = 1 - \frac{P}{u} + \frac{Q}{u - \kappa},$$

originally found for $\mathfrak{o}_N$ by [A. & Al. Zamolodchikov, 1979].

The operator Q is defined by the formulas

$$Q = \sum_{i,j=1}^{N} e_{ij} \otimes e_{i'j'} \qquad \text{and} \qquad Q = \sum_{i,j=1}^{N} \varepsilon_i \varepsilon_j \, e_{ij} \otimes e_{i'j'},$$

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We use the notation $i' = N - i + 1$, and set

$\varepsilon_i = 1$ for $i = 1, \ldots, n$ and

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The parameter κ is

$$\kappa = \begin{cases} N/2 - 1 & \text{for} \quad \mathfrak{o}_N \\ n + 1 & \text{for} \quad \mathfrak{sp}_{2n}. \end{cases}$$

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$$T(u) = \sum_{i,j=1}^{N} e_{ij} \otimes t_{ij}(u) \in \mathrm{End}\, V \otimes \mathrm{X}(\mathfrak{g})[[u^{-1}]]$$

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The defining relations take the form

$$\big[t_{ij}(u), t_{kl}(v)\big] = \frac{1}{u-v}\Big(t_{kj}(u)\, t_{il}(v) - t_{kj}(v)\, t_{il}(u)\Big).$$

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$$- \frac{1}{u-v-\kappa}\Big(\delta_{k i'} \sum_{p=1}^{N} \theta_{ip}\, t_{pj}(u)\, t_{p'l}(v) - \delta_{l j'} \sum_{p=1}^{N} \theta_{jp}\, t_{kp'}(v)\, t_{ip}(u)\Big),$$

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Definition. The Yangian in the R-matrix presentation is the algebra $\mathrm{Y}^R(\mathfrak{g})$ defined by

$$\mathrm{Y}^R(\mathfrak{g}) = \{y \in \mathrm{X}(\mathfrak{g}) \mid \mu_f(y) = y \quad \text{for all} \quad \mu_f\},$$

where the automorphism $\mu_f : \mathrm{X}(\mathfrak{g}) \to \mathrm{X}(\mathfrak{g})$ is defined by

$$\mu_f : T(u) \mapsto f(u)\,T(u), \qquad f(u) \in 1 + u^{-1}\mathbb{C}[[u^{-1}]].$$

Theorem [Wendlandt, 2017].

$$S^2\big(T(u)\big)\,T(u+c_{\mathfrak{g}}/2)^{-1}=z(u)\,1$$

for a series $z(u)=1+z_2\,u^{-2}+z_3\,u^{-3}+\dots\in \mathrm{X}(\mathfrak{g})[[u^{-1}]]$.

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$$\mathrm{X}(\mathfrak{g}) = \mathrm{ZX}(\mathfrak{g}) \otimes \mathrm{Y}^R(\mathfrak{g}).$$

We have the isomorphism $\mathrm{Y}^R(\mathfrak{g}) \cong \mathrm{Y}(\mathfrak{g})$,

$$\mathrm{X}(\mathfrak{g})/\langle z(u) = 1\rangle \cong \mathrm{Y}(\mathfrak{g}), \qquad T(u) \mapsto (\rho \otimes 1)\,\mathcal{R}(-u).$$

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In addition, in types B_n, C_n and D_n respectively set

$$\alpha_n = \epsilon_n, \qquad \alpha_n = 2\epsilon_n \qquad \text{and} \qquad \alpha_n = \epsilon_{n-1} + \epsilon_n,$$

Drinfeld presentation

Suppose that $\mathfrak{g}$ is a simple Lie algebra of rank n and let $A = [a_{ij}]$ be the associated Cartan matrix.

Let $\alpha_1, \dots, \alpha_n$ be the corresponding simple roots. Set

$$\alpha_i = \epsilon_i - \epsilon_{i+1}, \qquad i = 1, \dots, n-1, \qquad \text{for} \quad A_{n-1}.$$

In addition, in types B_n, C_n and D_n respectively set

$$\alpha_n = \epsilon_n, \qquad \alpha_n = 2\epsilon_n \qquad \text{and} \qquad \alpha_n = \epsilon_{n-1} + \epsilon_n,$$

Here $\epsilon_1, \dots, \epsilon_n$ is an orthonormal basis of an Euclidian space with the bilinear form $(.\,,.)$.

The Drinfeld Yangian $\mathrm{Y}^D(\mathfrak{g})$ is generated by elements $\kappa_{i\,r}$, $\xi_{i\,r}^{\pm}$

with $i = 1, \dots, n$ and $r = 0, 1, \dots$ subject to the defining relations

The Drinfeld Yangian $\mathrm{Y}^D(\mathfrak{g})$ is generated by elements $\kappa_{i\,r}$, $\xi_{i\,r}^{\pm}$ with $i = 1, \dots, n$ and $r = 0, 1, \dots$ subject to the defining relations

$$[\kappa_{i\,r}, \kappa_{j\,s}] = 0,$$

$$[\xi_{i\,r}^{+}, \xi_{j\,s}^{-}] = \delta_{ij}\, \kappa_{i\,r+s},$$

$$[\kappa_{i\,0}, \xi_{j\,s}^{\pm}] = \pm\, (\alpha_i, \alpha_j)\, \xi_{j\,s}^{\pm},$$

$$[\kappa_{i\,r+1}, \xi_{j\,s}^{\pm}] - [\kappa_{i\,r}, \xi_{j\,s+1}^{\pm}] = \pm\, \frac{(\alpha_i, \alpha_j)}{2} \big(\kappa_{i\,r}\, \xi_{j\,s}^{\pm} + \xi_{j\,s}^{\pm}\, \kappa_{i\,r}\big),$$

$$[\xi_{i\,r+1}^{\pm}, \xi_{j\,s}^{\pm}] - [\xi_{i\,r}^{\pm}, \xi_{j\,s+1}^{\pm}] = \pm\, \frac{(\alpha_i, \alpha_j)}{2} \big(\xi_{i\,r}^{\pm}\, \xi_{j\,s}^{\pm} + \xi_{j\,s}^{\pm}\, \xi_{i\,r}^{\pm}\big),$$

$$\sum_{p \in \mathfrak{S}_m} [\xi_{i\,r_{p(1)}}^{\pm}, [\xi_{i\,r_{p(2)}}^{\pm}, \dots, [\xi_{i\,r_{p(m)}}^{\pm}, \xi_{j\,s}^{\pm}] \dots]] = 0,$$

where the last relation holds for all $i \neq j$ with $m = 1 - a_{ij}$.

Combine the generators of $\mathrm{Y}^D(\mathfrak{g})$ into power series in u^{-1},

$$\kappa_i(u) = 1 + \sum_{r=0}^{\infty} \kappa_{i\,r}\, u^{-r-1} \qquad \text{and} \qquad \xi_i^{\pm}(u) = \sum_{r=0}^{\infty} \xi_{i\,r}^{\pm}\, u^{-r-1}$$

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Theorem [Drinfeld, 1988]. Every finite-dimensional irreducible representation L of the algebra $\mathrm{Y}^D(\mathfrak{g})$ contains a unique (up to constant factor) nonzero vector ζ (the highest vector) such that

$$\begin{aligned} \xi_i^{+}(u)\,\zeta &= 0, \\ \kappa_i(u)\,\zeta &= \frac{P_i(u+d_i)}{P_i(u)}\,\zeta, \qquad d_i = (\alpha_i, \alpha_i)/2, \end{aligned}$$

for $i = 1, \dots, n$, where $P_1(u), \dots, P_n(u)$ are monic polynomials.

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The construction is based on the natural embedding

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Main sticking point for types B, C, D:

There is no natural embedding of $\mathrm{X}(\mathfrak{g}_{N-2})$ into $\mathrm{X}(\mathfrak{g}_N)$.

Quasideterminants

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Consider a $k \times k$ matrix of the form

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Then its (k,k)-quasideterminant is defined by

$$\begin{vmatrix} A & B \\ C & \boxed{D} \end{vmatrix} = D - CA^{-1}B.$$

[Gelfand–Retakh, 1991].

Theorem [Jing–Liu–M., 2017]. The mapping

$$t_{ij}(u) \mapsto \begin{vmatrix} t_{11}(u) & t_{1j}(u) \\ t_{i1}(u) & \boxed{t_{ij}(u)} \end{vmatrix} = t_{ij}(u) - t_{i1}(u)\, t_{11}(u)^{-1}\, t_{1j}(u)$$

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It is consistent with the embedding $\mathfrak{g}_{N-2} \hookrightarrow \mathfrak{g}_N$.

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Theorem [JLM, 2017]. The mapping

$$\psi_m^{(N)} : t_{ij}(u) \mapsto \begin{vmatrix} t_{11}(u) & \dots & t_{1m}(u) & t_{1j}(u) \\ \dots & \dots & \dots & \dots \\ t_{m1}(u) & \dots & t_{mm}(u) & t_{mj}(u) \\ t_{i1}(u) & \dots & t_{im}(u) & \boxed{t_{ij}(u)} \end{vmatrix},$$

with $m+1 \leqslant i,j \leqslant (m+1)'$ defines an injective homomorphism

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with $m+1 \leqslant i,j \leqslant (m+1)'$ defines an injective homomorphism

$\mathrm{X}(\mathfrak{g}_{N-2m}) \to \mathrm{X}(\mathfrak{g}_N)$.

Moreover, we have the equality of maps

$$\psi_l^{(N)} \circ \psi_m^{(N-2l)} = \psi_{l+m}^{(N)}.$$

Gaussian generators

Apply the Gauss decomposition to the matrix $T(u)$,

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for uniquely determined matrices $F(u)$, $H(u)$, $E(u)$,

$$F(u) = \begin{bmatrix} 1 & \dots & 0 \\ \vdots & \ddots & \vdots \\ f_{N1}(u) & \dots & 1 \end{bmatrix}, \qquad E(u) = \begin{bmatrix} 1 & \dots & e_{1N}(u) \\ \vdots & \ddots & \vdots \\ 0 & \dots & 1 \end{bmatrix},$$

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and $H(u) = \operatorname{diag}\big[h_1(u), \dots, h_N(u)\big]$.

The entries of the matrices $F(u)$, $H(u)$, $E(u)$ are expressed in terms of quasideterminants of submatrices of $T(u)$ as follows.

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$$h_i(u) = \begin{vmatrix} t_{1\,1}(u) & \dots & t_{1\,i-1}(u) & t_{1\,i}(u) \\ \vdots & \ddots & \vdots & \vdots \\ t_{i-1\,1}(u) & \dots & t_{i-1\,i-1}(u) & t_{i-1\,i}(u) \\ t_{i\,1}(u) & \dots & t_{i\,i-1}(u) & \boxed{t_{i\,i}(u)} \end{vmatrix},$$

for $i = 1, \dots, N$.

Moreover, for $1 \leqslant i < j \leqslant N$ we have

$$e_{ij}(u) = h_i(u)^{-1} \begin{vmatrix} t_{1\,1}(u) & \dots & t_{1\,i-1}(u) & t_{1\,j}(u) \\ \vdots & \ddots & \vdots & \vdots \\ t_{i-1\,1}(u) & \dots & t_{i-1\,i-1}(u) & t_{i-1\,j}(u) \\ t_{i\,1}(u) & \dots & t_{i\,i-1}(u) & \boxed{t_{ij}(u)} \end{vmatrix}$$

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and

$$f_{ji}(u) = \begin{vmatrix} t_{1\,1}(u) & \dots & t_{1\,i-1}(u) & t_{1\,i}(u) \\ \vdots & \ddots & \vdots & \vdots \\ t_{i-1\,1}(u) & \dots & t_{i-1\,i-1}(u) & t_{i-1\,i}(u) \\ t_{j\,1}(u) & \dots & t_{j\,i-1}(u) & \boxed{t_{ji}(u)} \end{vmatrix} h_i(u)^{-1}.$$

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$$\kappa_i(u) = h_i\Big(u - \frac{i-1}{2}\Big)^{-1} h_{i+1}\Big(u - \frac{i-1}{2}\Big)$$

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for $i = 1, \dots, n-1$, and

$$\kappa_n(u) = \begin{cases} h_n\Big(u - \dfrac{n-1}{2}\Big)^{-1} h_{n+1}\Big(u - \dfrac{n-1}{2}\Big) & \text{for} \quad \mathfrak{o}_{2n+1} \\[2ex] h_n\Big(u - \dfrac{n}{2}\Big)^{-1} h_{n+1}\Big(u - \dfrac{n}{2}\Big) & \text{for} \quad \mathfrak{sp}_{2n} \\[2ex] h_{n-1}\Big(u - \dfrac{n-2}{2}\Big)^{-1} h_{n+1}\Big(u - \dfrac{n-2}{2}\Big) & \text{for} \quad \mathfrak{o}_{2n}. \end{cases}$$

Furthermore, for $i = 1, \ldots, n-1$ set

$$\xi_i^+(u) = f_{i+1\,i}\Big(u - \frac{i-1}{2}\Big), \qquad \xi_i^-(u) = e_{i\,i+1}\Big(u - \frac{i-1}{2}\Big),$$

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and

$$\xi_n^-(u) = \begin{cases} e_{n\, n+1}\Big(u - \dfrac{n-1}{2}\Big) & \text{for} \quad \mathfrak{o}_{2n+1} \\ 1/2\, e_{n\, n+1}\big(u - n/2\big) & \text{for} \quad \mathfrak{sp}_{2n} \\ e_{n-1\, n+1}\Big(u - \dfrac{n-2}{2}\Big) & \text{for} \quad \mathfrak{o}_{2n}. \end{cases}$$

Introduce elements of $\mathrm{X}(\mathfrak{g}_N)$ by the respective expansions into power series in u^{-1},

$$\kappa_i(u) = 1 + \sum_{r=0}^{\infty} \kappa_{i\,r}\, u^{-r-1} \qquad \text{and} \qquad \xi_i^{\pm}(u) = \sum_{r=0}^{\infty} \xi_{i\,r}^{\pm}\, u^{-r-1}$$

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for $i = 1, \dots, n$.

Theorem [JLM, 2017]. The mapping which sends the generators $\kappa_{i\,r}$ and $\xi_{i\,r}^{\pm}$ of $\mathrm{Y}^D(\mathfrak{g}_N)$ to the elements of $\mathrm{X}(\mathfrak{g}_N)$ with the same names defines an isomorphism $\mathrm{Y}^D(\mathfrak{g}_N) \cong \mathrm{Y}^R(\mathfrak{g}_N)$.

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Applications: coproduct and representations

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and the isomorphism $\mathrm{Y}^D(\mathfrak{g}_N) \cong \mathrm{Y}^R(\mathfrak{g}_N)$ can be used to calculate the coproduct in terms of the Drinfeld presentation (which has not been explicitly described).

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A representation V of the algebra $\mathrm{X}(\mathfrak{g}_N)$ is called a highest weight representation if there exists a nonzero vector $\xi \in V$ such that V is generated by ξ,

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for some formal series

$$\lambda_i(u) = 1 + \lambda_i^{(1)} u^{-1} + \lambda_i^{(2)} u^{-2} + \cdots, \qquad \lambda_i^{(r)} \in \mathbb{C}.$$

Every finite-dimensional irreducible representation of $\mathrm{X}(\mathfrak{g}_N)$ is isomorphic to the highest weight representation $L(\lambda(u))$ for a certain N-tuple of formal series $\lambda(u) = \big(\lambda_1(u), \dots, \lambda_N(u)\big)$ with

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Hence, by applying the isomorphism $\mathrm{Y}^D(\mathfrak{g}_N) \cong \mathrm{Y}^R(\mathfrak{g}_N)$ we thus obtain the Drinfeld classification theorem for finite-dimensional irreducible representations of $\mathrm{Y}^D(\mathfrak{g}_N)$.

Centers of the Yangians

The center $\mathrm{ZY}(\mathfrak{gl}_N)$ of the Yangian $\mathrm{Y}(\mathfrak{gl}_N)$ is generated by the coefficients of the quantum determinant

$$\mathrm{qdet}\, T(u) = \sum_{p \in \mathfrak{S}_N} \mathrm{sgn}\, p \cdot t_{p(1)\,1}(u+N-1) \dots t_{p(N)\,N}(u).$$

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The Wendlandt series $z(u)$ is given by

$$z(u)^{-1} = \frac{1}{N} \operatorname{tr} T(u+N)\, T(u)^{-1} = \frac{\operatorname{qdet} T(u+1)}{\operatorname{qdet} T(u)},$$

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the last equality is the quantum Liouville formula

[Nazarov, 1991].

The center $\mathrm{ZY}(\mathfrak{g}_N)$ of the extended Yangian $\mathrm{X}(\mathfrak{g}_N)$ is generated by the coefficients of the series

$$\zeta(u) = \frac{1}{N}\,\mathrm{tr}\,T(u+\kappa)'\,T(u), \qquad \kappa = N/2 \mp 1,$$

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