The University of Sydney

MATH2008 Introduction to Modern Algebra

(http://www.maths.usyd.edu.au/u/UG/IM/MATH2008/)

Semester2, 2003 — Lecturer: R. Howlett

Tutorial 4

In this tutorial $\mathcal{C}[a,b]$ denotes the inner product space of all continuous functions from $[a,b]$ to $\mathbb{R}$, with inner product given by $(f,g)=\int_a^b f(x)g(x)\,dx$.

1. Define $f,\ g\in\mathcal{C}[-1,1]$ by $f(x)=x$, $g(x)=x^3$. Work out $\|f\|$, $\|g\|$ and (f,g).

Solution.

$\|f\|=\sqrt{\int_{-1}^1 x^2\,dx}=\sqrt{\frac{1}{3}x^3\big]_{-1}^1}=\sqrt{2/3}$.

$\|g\|=\sqrt{\int_{-1}^1 x^6\,dx}=\sqrt{\frac{1}{7}x^7\big]_{-1}^1}=\sqrt{2/7}$.

$(f,g)=\int_{-1}^1 x\,x^3\,dx=\frac{1}{5}x^5\big]_{-1}^1=2/5$.

2. Show that 1 and x are orthogonal in $\mathcal{C}[-1,1]$.

Solution.

We must show that $(f,g)=0$, where f and g are defined by $f(x)=1$ and $g(x)=x$ (for all $x\in[-1,1]$). We have $(f,g)=\int_{-1}^1 x\,dx=\frac{1}{2}x^2\big]_{-1}^1=0$, as required.

3. For which values of k and m are the polynomial functions x^k and x^m orthogonal in $\mathcal{C}[-1,1]$?

Solution.

Let k and m be arbitrary nonnegative integers. Then

$$(x^k,x^m)=\int_{-1}^1 x^{k+m}\,dx=\frac{x^{k+m+1}}{k+m+1}\Big]_{-1}^1=\frac{1}{k+m+1}(1^{k+m+1}-(-1)^{k+m+1}),$$

which is 0 if and only if $k+m+1$ is even.

4. Prove the following properties of the inner product on $\mathcal{C}[-1,1]$.

(*i*) $(f,g)=(g,f)$ for all $f,\ g\in\mathcal{C}[-1,1]$.

(*ii*) $(f+g,h)=(f,h)+(g,h)$ for all $f,\ g,\ h\in\mathcal{C}[-1,1]$.

(*iii*) $(kf,g)=k(f,g)$ for all $k\in\mathbb{R}$ and all $f,\ g\in\mathcal{C}[-1,1]$.

Solution.

Let f, g and h be arbitrary elements of $\mathcal{C}[-1,1]$, and k an arbitrary real number. Then

(*i*) $(f,g)=\int_{-1}^1 f(x)g(x)\,dx=\int_{-1}^1 g(x)f(x)\,dx=(g,f)$;

(*ii*)
$$\begin{aligned}(f+g,h)&=\int_{-1}^1(f(x)+g(x))h(x)\,dx\\&=\int_{-1}^1 f(x)h(x)\,dx+\int_{-1}^1 g(x))h(x)\,dx\\&=(f,h)+(g,h);\end{aligned}$$

(*iii*)
$$\begin{aligned}(kf,g)&=\int_{-1}^1 kf(x)g(x)\,dx\\&=k\int_{-1}^1 f(x)g(x)\,dx\\&=k(f,g).\end{aligned}$$

Hence the stated properties hold for all $f,\ g,\ h\in\mathcal{C}[-1,1]$ and $k\in\mathbb{R}$.

5. Consider the inner product space $\mathcal{C}[1,2]$.

(*i*) Apply the Gram-Schmidt process to the set $\{1,x,x^2\}$ to produce an orthogonal set.

(*ii*) Using the results of Part (*i*), find the parabola that best approximates $\ln x$ over $[1,2]$. (That is, project $\ln x$ onto the subspace spanned by $\{1,x,x^2\}$.)

Solution.

(*i*) Define $f_0,\ f_1,\ f_2\in\mathcal{C}[1,2]$ by $f_0(x)=1$, $f_1(x)=x$ and $f_2(x)=x^2$ (for all $x\in\mathcal{C}[1,2]$). Since $\int_1^2 x^k\,dx=\frac{2^{k+1}-1}{k+1}$, we find that $(f_0,f_0)=1$, $(f_0,f_1)=3/2$, $(f_1,f_1)=(f_0,f_2)=7/3$, and $(f_1,f_2)=15/4$. Applying the Gram-Schmidt process to the basis $\{f_0,f_1,f_2\}$ yields $\{g_0,g_1,g_2\}$, where

$$\begin{aligned}g_0&=f_0;\\g_1&=f_1-\frac{(f_1,g_0)}{(g_0,g_0)}g_0;\\g_2&=f_2-\frac{(f_2,g_0)}{(g_0,g_0)}g_0-\frac{(f_2,g_1)}{(g_1,g_1)}g_1.\end{aligned}$$

This immediately gives $g_1=f_1-\frac{3}{2}f_0$, and we then find that

$$\begin{aligned}(f_2,g_1)&=(f_2,f_1)-\tfrac{3}{2}(f_2,f_0)=\tfrac{15}{4}-\tfrac{7}{2}=\tfrac{1}{4}\\(g_1,g_1)&=(g_1,f_1-\tfrac{3}{2}g_0)=(g_1,f_1)\\&=(f_1,f_1)-\tfrac{3}{2}(f_0,f_1)=\tfrac{7}{3}-\tfrac{9}{4}=\tfrac{1}{12}.\end{aligned}$$

Hence

$$\begin{aligned}g_2&=f_2-\tfrac{7}{3}f_0-\tfrac{1/4}{1/12}(f_1-\tfrac{3}{2}f_0)\\&=f_2-\tfrac{7}{3}f_0-3(f_1-\tfrac{3}{2}f_0)=f_2-3f_1+\tfrac{13}{6}f_0.\end{aligned}$$

Thus $g_0(x) = 1$, $g_1(x) = x - \frac{3}{2}$ and $g_2(x) = x^2 - 3x + \frac{13}{6}$.

(ii) The projection is $p = \frac{(\ln, g_0)}{(g_0,g_0)} g_0 + \frac{(\ln, g_1)}{(g_1,g_1)} g_1 + \frac{(\ln x, g_2)}{(g_2,g_2)} g_2$. Using integration by parts, or by consulting a list of standard integrals, we find that

$$\int x^n \ln x \, dx = \frac{x^{n+1}}{n+1}\Big(\ln x - \frac{1}{n+1}\Big),$$

and so for each $n = 1$, 2 or 3,

$$\begin{aligned}(\ln, f_n) &= \int_1^2 x^n \ln x \, dx \\ &= \tfrac{2^{n+1}}{n+1}\big(\ln 2 - \tfrac{1}{n+1}\big) - \tfrac{1}{n+1}\big(\ln 1 - \tfrac{1}{n+1}\big) = \tfrac{2^{n+1}}{n+1}\ln 2 - \tfrac{2^{n+1}-1}{(n+1)^2}\end{aligned}$$

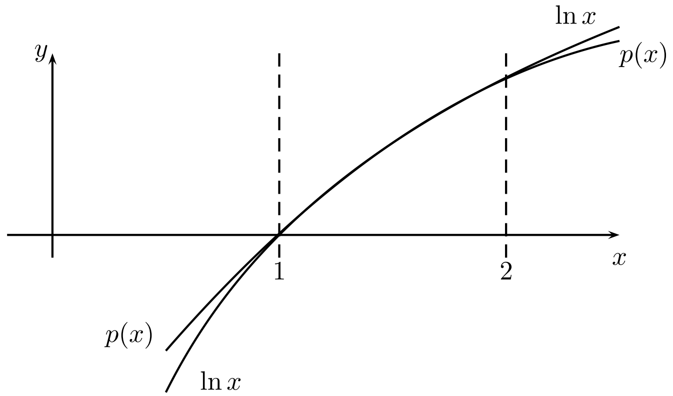
since $\ln 1 = 0$. Thus

$$\begin{aligned}(\ln, g_0) &= (\ln, f_0) = 2\ln 2 - 1 \\ (\ln, g_1) &= (\ln, f_1) - \tfrac{3}{2}(\ln, f_0) \\ &= (2\ln 2 - \tfrac{3}{4}) - \tfrac{3}{2}(2\ln 2 - 1) \\ &= -\ln 2 + \tfrac{3}{4} \\ (\ln, g_2) &= (\ln, f_2) - 3(\ln, f_1) + \tfrac{13}{3}(\ln, f_0) \\ &= (\tfrac{8}{3}\ln 2 - \tfrac{7}{9}) - 3(2\ln 2 - \tfrac{3}{4}) + \tfrac{13}{6}(2\ln 2 - 1) \\ &= \ln 2 - \tfrac{25}{36}.\end{aligned}$$

We found in Part (i) above that $(g_0, g_0) = 1$ and $(g_1, g_1) = \frac{1}{12}$, and we also have that

$$\begin{aligned}(g_2, g_2) &= (g_2, f_2 - \frac{(f_2, g_0)}{(g_0, g_0)} g_0 - \frac{(f_2, g_1)}{(g_1, g_1)} g_1) \\ &= (g_2, f_2) \\ &= (f_2 - 3f_1 + \tfrac{13}{6} f_0, f_2) \\ &= (f_2, f_2) - 3(f_1, f_2) + \tfrac{13}{6}(f_0, f_2) \\ &= \tfrac{31}{5} - \tfrac{45}{4} + \tfrac{91}{18} = \tfrac{1}{180}.\end{aligned}$$

So we obtain $p = (2\ln 2 - 1)g_0 + 12(-\ln 2 + \frac{3}{4})g_1 + 180(\ln 2 - \frac{25}{36})g_2$. (This gives $p(x) = (180\ln 2 - 125)x^2 + (384 - 552\ln 2)x + (410\ln 2 - \frac{856}{3})$. According to magma, the distance from ln to p is about 0.0020333.)



6. Compute the 3rd degree Legendre polynomial. (That is, apply Gram-Schmidt to $\{1, x, x^2, x^3\}$, working in $\mathcal{C}[-1, 1]$.)

Solution.

Let $\{g_0, g_1, g_2, g_3\}$ be the basis given by Gram-Schmidt. Then $g_0 = 1$, and since $(1, x) = \int_{-1}^{1} x\, dx = 0$ we find that $g_1 = x$. Next, $(x^2, 1) = \frac{2}{3}$, $(1, 1) = 2$ and $(x^2, x) = 0$, giving $g_2 = x^2 - \frac{1}{3}$. And $(x^3, g_0) = (x^3, g_2) = 0$, so that $g_3 = x^3 - \frac{(x^3, x)}{(x,x)} x = x^3 - \frac{2/5}{2/3} x = x^3 - \frac{3}{5} x$.

7. In $\mathcal{C}[0, 2\pi]$, show that $\{1, \sin x, \cos x\}$ is an orthogonal set. Find the lengths in $\mathcal{C}[0, 2\pi]$ of each of 1, $\sin x$, $\cos x$.

Solution.

$$\begin{aligned}(1, \sin) &= \int_0^{2\pi} \sin x \, dx = -\cos x\Big]_0^{2\pi} = 0; \\ (1, \cos) &= \int_0^{2\pi} \cos x \, dx = \sin x\Big]_0^{2\pi} = 0; \\ (\sin, \cos) &= \int_0^{2\pi} \sin x \, \cos x \, dx = \frac{1}{2}\int_0^{2\pi} \sin 2x \, dx = -\frac{1}{4}\cos 2x\Big]_0^{2\pi} = 0.\end{aligned}$$

This proves orthogonality. Clearly $(1, 1) = \int_0^{2\pi} 1 \, dx = 2\pi$, and

$$\begin{aligned}(\cos, \cos) &= \int_0^{2\pi} \cos^2 x \, dx = \frac{1}{2}\int_0^{2\pi} (1 + \cos 2x)\, dx = \tfrac{x}{2} + \tfrac{1}{4}\sin 2x\Big]_0^{2\pi} = \pi, \\ (\sin, \sin) &= \int_0^{2\pi} \sin^2 x \, dx = \frac{1}{2}\int_0^{2\pi} (1 - \cos 2x)\, dx = \tfrac{x}{2} - \tfrac{1}{4}\sin 2x\Big]_0^{2\pi} = \pi.\end{aligned}$$

Thus $\|\cos\| = \sqrt{(\cos, \cos)} = \sqrt{\pi}$ and $\|\sin\| = \sqrt{\pi}$, while $\|1\| = \sqrt{2\pi}$.

8. In $\mathcal{C}[0, 2\pi]$, find the projection of the function x onto the subspace spanned by $\{1, \sin x, \cos x\}$.

Solution.

The projection p is given by $p(x) = \frac{(x,1)}{(1,1)} 1 + \frac{(x,\sin)}{(\sin,\sin)} \sin x + \frac{(x,\cos)}{(\cos,\cos)} \cos x$. Now $(x, \sin) = \sin x - x\cos x\Big]_0^{2\pi} = -2\pi$ and $(x, \cos) = x \sin x - \cos x\Big]_0^{2\pi} = 0$, while $(x, 1) = 2\pi^2$. So we conclude that $p(x) = \pi - 2\sin x$.

9. Let V be an inner product space, and $v \in V$. Prove that $(\underset{\sim}{0}, \underset{\sim}{v}) = (\underset{\sim}{v}, \underset{\sim}{0}) = 0$.

Solution.

One of the inner product axioms says that $(k\underset{\sim}{x}, \underset{\sim}{y}) = k(\underset{\sim}{x}, \underset{\sim}{y})$, for all vectors $\underset{\sim}{x}$, $\underset{\sim}{y}$ and all scalars k. Apply this with $\underset{\sim}{x} = \underset{\sim}{0}$ (the zero vector), $k = 0$ (the zero scalar) and $\underset{\sim}{y} = \underset{\sim}{v}$. Since $0\underset{\sim}{0} = \underset{\sim}{0}$ holds in any vector space—indeed, $0\underset{\sim}{u} = \underset{\sim}{0}$ for all $\underset{\sim}{u}$—we deduce that $(\underset{\sim}{0}, \underset{\sim}{v}) = (0\underset{\sim}{0}, \underset{\sim}{v}) = 0(\underset{\sim}{0}, \underset{\sim}{v}) = 0$, as required.