The University of Sydney

MATH2902 Vector Spaces

(http://www.maths.usyd.edu.au/u/UG/IM/MATH2902/)

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Tutorial 9

1. Compute the given products of permutations.

(*i*) $\begin{bmatrix} 1 & 2 & 3 & 4 \\ 2 & 1 & 4 & 3 \end{bmatrix} \begin{bmatrix} 1 & 2 & 3 & 4 \\ 2 & 3 & 4 & 1 \end{bmatrix}$ (*ii*) $\begin{bmatrix} 1 & 2 & 3 & 4 & 5 \\ 4 & 1 & 5 & 2 & 3 \end{bmatrix} \begin{bmatrix} 1 & 2 & 3 & 4 & 5 \\ 5 & 4 & 3 & 2 & 1 \end{bmatrix}$

(*iii*) $\left(\begin{bmatrix} 1 & 2 & 3 & 4 \\ 2 & 1 & 4 & 3 \end{bmatrix} \begin{bmatrix} 1 & 2 & 3 & 4 \\ 2 & 3 & 4 & 1 \end{bmatrix} \right) \begin{bmatrix} 1 & 2 & 3 & 4 \\ 4 & 3 & 1 & 2 \end{bmatrix}$

(*iv*) $\begin{bmatrix} 1 & 2 & 3 & 4 \\ 2 & 1 & 4 & 3 \end{bmatrix} \left(\begin{bmatrix} 1 & 2 & 3 & 4 \\ 2 & 3 & 4 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 & 3 & 4 \\ 4 & 3 & 1 & 2 \end{bmatrix} \right)$

Solution.

Remember that permutations are functions, and multiplication of permutations is composition of functions. If we let the first factor in part (*i*) be σ and the second τ then we have $\tau(1) = 2$ and $\sigma(2) = 1$; so $(\sigma\tau)(1) = \sigma(\tau(1)) = \sigma(2) = 1$. Similarly,

$$\begin{aligned} (\sigma\tau)(2) &= \sigma(\tau(2)) = \sigma(3) = 4, \\ (\sigma\tau)(3) &= \sigma(\tau(3)) = \sigma(4) = 3, \\ (\sigma\tau)(4) &= \sigma(\tau(4)) = \sigma(1) = 2. \end{aligned}$$

See also the examples on page 173 of [VST].

The answers are as follows:

(*i*) $\begin{bmatrix} 1 & 2 & 3 & 4 \\ 1 & 4 & 3 & 2 \end{bmatrix}$ (*ii*) $\begin{bmatrix} 1 & 2 & 3 & 4 & 5 \\ 3 & 2 & 5 & 1 & 4 \end{bmatrix}$

(*iii*) $\begin{bmatrix} 1 & 2 & 3 & 4 \\ 2 & 3 & 1 & 4 \end{bmatrix}$ (*iv*) $\begin{bmatrix} 1 & 2 & 3 & 4 \\ 2 & 3 & 1 & 4 \end{bmatrix}$.

2. Calculate the parity of each permutation appearing in Exercise 1.

Solution.

Recall that the "length" of a permutation is obtained by counting the total number of instances of a number in the second row being larger than a number to its right. In part (*i*) the first factor is even (its length is 2) and the other is odd (length 3). Their product is odd (length 3). In part (*ii*) the first factor has length 5 and is therefore odd, the second has length 10 (the maximum possible for a

permutation of $\{1, 2, 3, 4, 5\}$) and is therefore even. The product must be odd. In part (*iii*) the factors are even, odd and odd.

3. Use row and column operations to calculate the determinant of

$$\begin{pmatrix} 1 & 5 & 11 & 2 \\ 2 & 11 & -6 & 8 \\ -3 & 0 & -452 & 6 \\ -3 & -16 & -4 & 13 \end{pmatrix}$$

Solution.

$$\begin{pmatrix} 1 & 5 & 11 & 2 \\ 2 & 11 & -6 & 8 \\ -3 & 0 & -452 & 6 \\ -3 & -16 & -4 & 13 \end{pmatrix} \xrightarrow[R_4:=R_4+3R_1]{\substack{R_2:=R_2-2R_1 \\ R_3:=R_3+3R_1}} \begin{pmatrix} 1 & 5 & 11 & 2 \\ 0 & 1 & -28 & 4 \\ 0 & 15 & -419 & 12 \\ 0 & -1 & 29 & 19 \end{pmatrix}$$

$$\xrightarrow[R_4:=R_4+R_2]{R_3:=R_3-15R_2} \begin{pmatrix} 1 & 5 & 11 & 2 \\ 0 & 1 & -28 & 4 \\ 0 & 0 & 1 & -48 \\ 0 & 0 & 1 & 23 \end{pmatrix}$$

$$\xrightarrow{R_4:=R_4-R_3} \begin{pmatrix} 1 & 5 & 11 & 2 \\ 0 & 1 & -28 & 4 \\ 0 & 0 & 1 & -48 \\ 0 & 0 & 0 & 71 \end{pmatrix}.$$

We have only used row operations of the kind $R_i := R_i + \alpha R_j$, and these do not change the determinant. Now performing lots of obvious column operations, where we just add multiples of columns to other columns, produces the diagonal matrix with entries 1, 1, 1, 71. So the determinant is 71.

4. For each permutation $\sigma \in S_n$ define P_σ to be the $n \times n$ matrix with (i, j)-entry equal to 1 if $i = \sigma(j)$ and 0 if $i \neq \sigma(j)$. Prove that $P_\sigma P_\tau = P_{\sigma\tau}$ for all $\sigma, \tau \in S_n$.

Solution.

It is convenient to use the Kronecker delta, defined by

$$\delta_{ij} = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{if } i \neq j. \end{cases}$$

If we let the $(i, j)^{\text{th}}$ entries of P_σ and P_τ be (respectively) s_{ij} and t_{ij} then we have $s_{ij} = \delta_{i\,\sigma(j)}$ and $t_{ij} = \delta_{i\,\tau(j)}$. Hence the $(i, j)^{\text{th}}$ entry of $P_\sigma P_\tau$ is $\sum_{k=1}^n s_{ik}t_{kj} = \sum_{k=1}^n \delta_{i\,\sigma(k)}\delta_{k\,\tau(j)}$. For $k = \tau(j)$ we have $\delta_{k\,\tau(j)} = 1$, and hence $\delta_{i\,\sigma(k)}\delta_{k\,\tau(j)} = \delta_{i\,\sigma(k)} = \delta_{i\,\sigma(\tau(j))}$. Since $\delta_{k\,\tau(j)} = 0$ when $k \neq \tau(j)$ the terms in the sum corresponding to the other values of k are zero. Hence the $(i, j)^{\text{th}}$ entry of ST is 1 if i equals $\sigma(\tau(j)) = (\sigma\tau)(j)$, and 0 otherwise.

5. What is the determinant of the matrix P_σ defined in Exercise 4?

Solution.

This is one of the rare cases when it is quite easy to work directly from the definition of the determinant as given in 8.12 of [VST]. Let s_{ij} be the (i,j)-entry of P_σ, so that $s_{ij} = \delta_{i\sigma(j)}$. For any $\tau \in S_n$ the product $s_{1\tau(1)}s_{2\tau(2)}\ldots s_{n\tau(n)}$ will be zero unless $s_{i\,\tau(i)} \neq 0$ for all i; that is, unless $\delta_{i\,\sigma(\tau(i))} \neq 0$ for all i. So only for $\tau = \sigma^{-1}$ is the product nonzero. When $\tau = \sigma^{-1}$ the product is just $\delta_{11}\delta_{22}\ldots\delta_{nn} = 1$. Hence by Definition 8.12 we have

$$\det P_\sigma = \sum_{\tau \in S_n} \varepsilon(\tau)s_{1\tau(1)}s_{2\tau(2)}\ldots s_{n\tau(n)} = \varepsilon(\sigma^{-1})$$

which is equal to $\varepsilon(\sigma)$.

6. Consider the determinant

$$\det\begin{pmatrix} 1 & x_1 & x_1^2 & \ldots & x_1^{n-1} \\ 1 & x_2 & x_2^2 & \ldots & x_2^{n-1} \\ \vdots & \vdots & \vdots & & \vdots \\ 1 & x_n & x_n^2 & \ldots & x_n^{n-1} \end{pmatrix}.$$

Use row and column operations to evaluate this in the case $n = 3$. Then do the case $n = 4$. Then do the general case. (The answer is $\prod_{i>j}(x_i - x_j)$.)

Solution.

First subtract the first row from all the others. This does not alter the determinant. Then subtract x_1 times the first column from the second, x_1^2 times the first column from the third, $\ldots$, x_1^{n-1} times the first column from the n^{th}. This also leaves the determinant unchanged, and so the original determinant is equal to

$$\det\begin{pmatrix} x_2 - x_1 & x_2^2 - x_1^2 & \ldots & x_2^{n-1} - x_1^{n-1} \\ x_3 - x_1 & x_3^2 - x_1^2 & \ldots & x_3^{n-1} - x_1^{n-1} \\ \vdots & \vdots & & \vdots \\ x_n - x_1 & x_n^2 - x_1^2 & \ldots & x_n^{n-1} - x_1^{n-1} \end{pmatrix}.$$

We can take out factors of $x_2 - x_1$ from the first row, $x_3 - x_1$ from the second, $\ldots$, $x_n - x_1$ from the last, so that our determinant is equal to

$$(x_2 - x_1)(x_3 - x_1)\ldots(x_n - x_1)D,$$

where

$$D = \det\begin{pmatrix} 1 & x_2 + x_1 & x_2^2 + x_2x_1 + x_1^2 & \ldots & \sum_{j=0}^{n-2} x_2^{n-2-j}x_1^j \\ 1 & x_3 + x_1 & x_3^2 + x_3x_1 + x_1^2 & \ldots & \sum_{j=0}^{n-2} x_3^{n-2-j}x_1^j \\ \vdots & \vdots & \vdots & & \vdots \\ 1 & x_n + x_1 & x_n^2 + x_nx_1 + x_1^2 & \ldots & \sum_{j=0}^{n-2} x_n^{n-2-j}x_1^j \end{pmatrix}.$$

Next subtract x_1^i times the first column of D from the $i+1^{\text{th}}$ column, for $i = 1, 2, \ldots , n-2$, then subtract x_1^i times the second column from the $i+2^{\text{th}}$ column, for $i = 1, 2, \ldots , n-3$, and then x_1^i times the third column from the $i+3^{\text{th}}$, and so on. This eliminates x_1 altogether, and the given determinant equals

$$(x_2 - x_1)(x_3 - x_1)\ldots(x_n - x_1)\det\begin{pmatrix} 1 & x_2 & \ldots & x_2^{n-2} \\ 1 & x_3 & \ldots & x_3^{n-2} \\ \vdots & \vdots & & \vdots \\ 1 & x_n & \ldots & x_n^{n-2} \end{pmatrix}.$$

Repeating the steps to eliminate successively x_2, x_3, $\ldots$, yields the formula $\prod_{i>j}(x_i - x_j)$.

7. Let $p(x) = a_0 + a_1x + a_2x^2$, $q(x) = b_0 + b_1x + b_2x^2$, $r(x) = c_0 + c_1x + c_2x^2$. Prove that

$$\det\begin{pmatrix} p(x_1) & q(x_1) & r(x_1) \\ p(x_2) & q(x_2) & r(x_2) \\ p(x_3) & q(x_3) & r(x_3) \end{pmatrix} = (x_2 - x_1)(x_3 - x_1)(x_3 - x_2)\det\begin{pmatrix} a_0 & b_0 & c_0 \\ a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \end{pmatrix}.$$

Solution.

This can be done by using row and column operations in a similar fashion to Exercise 6. Alternatively, observe that

$$\begin{pmatrix} p(x_1) & q(x_1) & r(x_1) \\ p(x_2) & q(x_2) & r(x_2) \\ p(x_3) & q(x_3) & r(x_3) \end{pmatrix} = \begin{pmatrix} 1 & x_1 & x_1^2 \\ 1 & x_2 & x_2^2 \\ 1 & x_3 & x_3^2 \end{pmatrix}\begin{pmatrix} a_0 & b_0 & c_0 \\ a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \end{pmatrix}$$

so that the result follows form Exercise 3 and the fact that $\det A \det B = \det AB$ for all $n \times n$ matrices A and B.