THE UNIVERSITY OF SYDNEY

MATH2902 Vector Spaces

(http://www.maths.usyd.edu.au/u/UG/IM/MATH2902/)

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Tutorial 6

1. Let V be a real inner product space and v, $w \in V$.

(i) Use calculus to prove that the minimum value of $\langle v - \lambda w, v - \lambda w\rangle$ occurs at $\lambda = \langle v, w\rangle / \langle w, w\rangle$.

(ii) Put $\lambda = \langle v, w\rangle / \langle w, w\rangle$ and use $\langle v - \lambda w, v - \lambda w\rangle \geq 0$ to prove the Cauchy-Schwarz inequality (see p.105 of the book).

2. (i) Prove that the following four vectors form an orthonormal subset of R^4:

$$\frac{1}{2}\begin{pmatrix}1\\1\\1\\1\end{pmatrix}, \quad \frac{1}{2}\begin{pmatrix}1\\1\\-1\\-1\end{pmatrix}, \quad \frac{1}{2}\begin{pmatrix}1\\-1\\1\\-1\end{pmatrix}, \quad \frac{1}{2}\begin{pmatrix}1\\-1\\-1\\1\end{pmatrix}.$$

(ii) Express $\begin{pmatrix}5\\-2\\4\\-1\end{pmatrix}$ as a linear combination of the vectors in part (i).

3. Let A be a real $n \times n$ matrix which is symmetric (${}^{\mathrm{t}}A = A$). We say that A is *positive definite* if $({}^{\mathrm{t}}v)Av > 0$ for all nonzero $v \in \mathbb{R}^n$.

(i) Prove that $\langle u, v\rangle = ({}^{\mathrm{t}}u)Av$ defines an inner product on $\mathbb{R}^n$ if and only if A is symmetric and positive definite.

(ii) Prove that a diagonal matrix $D \in \mathrm{Mat}(n \times n, \mathbb{R})$ is positive definite if and only if all the diagonal entries of D are positive.

(iii) Prove that if $A = {}^{\mathrm{t}}TDT$ where T is invertible and D is positive definite then A is positive definite.

4. Show that $T \in \mathrm{Mat}(n \times n, \mathbb{R})$ has the property that ${}^{\mathrm{t}}TT = I$ if and only if the columns of T form an orthonormal basis of $\mathbb{R}^n$. Show that the matrix $T = \begin{pmatrix}1/\sqrt{3} & 1/\sqrt{2} & 1/\sqrt{6}\\ 1/\sqrt{3} & 0 & -2/\sqrt{6}\\ 1/\sqrt{3} & -1/\sqrt{2} & 1/\sqrt{6}\end{pmatrix}$ has this property, and that all of its columns are eigenvectors for the matrix $A = \begin{pmatrix}2&1&1\\1&2&1\\1&1&2\end{pmatrix}$. Use the previous exercise to show that A is positive definite.

5. Let $v_1 = \begin{pmatrix}6\\2\\3\end{pmatrix}$, $v_2 = \begin{pmatrix}3\\8\\5\end{pmatrix}$, $v_3 = \begin{pmatrix}1\\1\\11\end{pmatrix}$. Find u_1, u_2, u_3 which form an orthogonal basis of $\mathbb{R}^3$ and satisfy $u_1 = v_1$ and $\mathrm{Span}(u_1, u_2) = \mathrm{Span}(v_1, v_2)$.