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Modern Probabilistic Concepts in the Work of E. Abbe and A. De Moivre

E. Seneta

Department of Mathematical Statistics,
University of Sydney, Sydney, NSW 2006, Australia

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Abstract

The concepts to be treated are non-normal limit laws, and martingales. The historical role of the Poisson as a limiting distribution is discussed with emphasis on the Poisson approximation to the binomial. The Poisson is shown to figure in this last context in two works of Ernst Abbe (1878), (1895), within the context of counting with a haemacytometer, thus anticipating Student's work in the area. Abbe, whose name has recently been of interest in a time-series context, further obtained the standard normal limit for a standardized Poisson random variable and discussed the degree of approximation in both his limit laws. An indication of the implicit use by De Moivre in 1711 of a martingale notion in his treatment of the gambler's ruin problem is pointed out.

1. The Poisson approximation to the binomial: background

The question of the origin of the Poisson approximation to the binomial, which is the forerunner of general theorems on triangular arrays of random variables independent within rows in relation to convergence to a compound Poisson distribution, has some interesting features. This origin is usually perceived in the paper of Student (1907) which certainly contains a derivation (p. 353); and contains also the famous data-set of 400 independent readings (on the number of yeast cells in 400 squares of a haemacytometer) from (accordingly) a Poisson distribution with mean λ. This parameter is estimated by the sample mean, and a chi-squared goodness-of-fit applied to obtained and expected values. The other well-known source popularizing the Poisson distribution as the 'law of small numbers' (or the law of rare events) is the book of Bortkiewicz (1898) which appeared under this name. This contains the famous data on deaths (from horsekicks)/year/corps in the Prussian army, which is shown to be fitted extremely well if treated as a sample from the Poisson distribution.

Indeed, Poisson (1837) did not arrive at the Poisson as a limit of the binomial. As Ulbricht (1980) (see also Stigler (1982)) points out, Poisson begins with the (Pascal) probability distribution (Seneta (1979)) with probability mass function $\binom{m+x-1}{x}p^m q^x$, $x=0,1,2,\ldots$ for the number of Bernoulli trials, x, beyond the mth to attain m successes, where p is the success probability in a single trial. Working with the summands of its distribution function (he seems to have been the first to use it, according to Sheynin (1978)) he considers the consequences of letting $m\to\infty$ and $q\to 0$

in such a way that $qm \to \lambda > 0$, which results in the Poisson expression $e^{-\lambda}\lambda^x/x!$, $x = 0, 1, 2, \ldots$ for the mass function. In fact even the appearance of the Poisson mass function can be traced to earlier sources: for example, M. G. Kendall (1968) points out that it occurs (with $\lambda = 1$) as the limiting form of the probability of x coincidences in the game of *rencontre* with two identical sets of n objects (first considered by Montmort *circa* 1708), as $n \to \infty$ in 1819 in the published writings of Thomas Young. In fact, Poisson limits of the first few binomial probabilities occur in the context of a special gambling problem in the writings of De Moivre†, and he states

'... the law of continuation of these equations is manifest.'

(See De Moivre (1967), Problem V, p. 45; David (1962), pp. 168–169.)

Given the preceding information the following footnote by the erudite A. A. Chuprov (1914) from an oration on the anniversary of the law of large numbers, which appears on p. 181 of a recent English translation of the Markov–Chuprov correspondence (Ondar (1981)), proved tantalizing:

'Apart from its methodological interest, the problem of counting the number of blood corpuscles deserves to be mentioned in the history of theroretical statistics because, in 1878, in studying it, the well known physicist Abbe developed the mathematical theory of the law of large numbers in the case of very small probabilities. His attention had been called to this problem by the manufacture, at the Zeiss factory, of a special instrument for counting blood corpuscles ... Abbe's paper completely escaped Bortkiewicz's attention and it was not mentioned by physicists, but Abbe's formulas found application both in the field for which they were developed and in research on plankton.'

Ernst Abbe had come to the attention of historians of statistics in 1966 and 1971 when O. B. Sheynin, followed by M. G. Kendall, described his remarkable discovery in 1863 of the distribution of the *first circular serial correlation coefficient* (see Seneta (1982)). Chuprov's footnote (above) seemed the more mysterious in that M. G. Kendall (1971) remarked:

'I cannot find that Abbe ever returned to the subject of the dissertation, but he was never eager to publish his work and his private papers, if they still exist, might possibly contain something of interest.'

Further investigation into Chuprov's writings (Tschuprow (1926)) revealed the following comment (given here in English translation from the Russian version of a Swedish oration)

'... Abbe encountered the same statistical-mathematical problem [as Poisson], when on direction from the Zeiss workshops he began to work on a projected haemacytometer. Abbe's formulae found application also in the study of plankton, but his contributions were not noticed outside this framework. About 20 years later Professor Bortkiewicz published his well-known investigation on "the law of small numbers" ... Subsequently, the same problem became an object of study of physicists, ignorant of the works of Abbe, Bortkiewicz, and others. Also, the talented disciple of Pearson, using the pseudonym "Student", published ... a short, but complete work on the topic developed by Abbe—the counting of haemacytes—with no concept of any of his predecessors.'

The reference to the Poisson distribution in the work of physicists may have in mind in

† The author is indebted to Chris Heyde for pointing this out.

particular the work of Smoluchowski in about 1915, in which it occurs in connection with a simple Galton–Watson process with immigration. However, again Chuprov gives no explicit citation to Abbe's investigations. These, however, exist as publications, and occur in Abbe's collected works (Abbe (1904–1940)).

The author is indebted to Dr W. Pfeiffer, of Carl Zeiss, Oberkochen, for, at his request, finding them and supplying copies. The following section supplements an encyclopedia entry on Abbe (Seneta (1982)).

2. Limit theorems in the work of Abbe

There appear to be two contributions pertaining to the material described by Chuprov: Abbe (1878) and Abbe (1895).

The first is largely descriptive, dealing firstly with the use of the haemacytometer constructed by C. Zeiss and secondly giving a theoretical discussion on the question of the degree of reliability with which the mean value (λ, say) of the number of blood corpuscles in a specific (say unit) volume can be determined by the method of counting. Here the Poisson expression $e^{-\lambda}\lambda^x/x!$, $x = 0, 1, 2, \ldots$ is given for the 'relative frequency' with which x corpuscles occur in such a volume. As an attempt to quantify the variability of such a Poisson random variable X about its mean λ, he states, again without proof, that, in effect, for large λ ('larger than 30 gives a quite adequate approximation') $(X-\lambda)/\sqrt{\lambda}$ has a standard normal distribution approximately.

Consequently, the 'probable' deviation from λ is $w = 0{\cdot}674\sqrt{\lambda}$ (in modern usage $2 \times 0{\cdot}674\sqrt{\lambda}$ is the interquartile range) which Abbe takes as the degree of reliability of the determination of λ. In these passages Abbe's work shows a degree of statistical confusion; he proposes, for example, also the probable deviation $\omega = w/\lambda = 0{\cdot}674/\sqrt{\lambda}$ corresponding to the 'relative' error $(X-\lambda)/\lambda$. What would seem to be most appropriate is a 50 per cent symmetric confidence interval for λ, on the basis of a sample $X_1, X_2, \ldots, X_n$, from a Poisson (λ) distribution about the sample mean $\bar{X}$, viz. $\bar{X} \pm 0{\cdot}674(\sqrt{\lambda/n})$, where λ need not be large, but n should. Indeed Student ((1907), p. 355) gives the expression $0{\cdot}674\,49\sqrt{\lambda/n}$ as the probable error of the mean.

The second paper (only the book which contains it suggests some connection with plankton) contains derivations of the two limit theorems implicit in the above (the Poisson approximation to the binomial, the normal approximation to the Poisson), both in local-limit form, starting with the binomial and Poisson mass functions respectively and considering their asymptotic behaviour. An important feature is that Abbe considers the *degree of approximation* by the limiting distribution also. Thus, proceeding from the Bin(n, p) distribution, where $np = \lambda =$ constant, he obtains, for large n, the approximate expression for the probability of value x:

$$(e^{-\lambda}\lambda^x/x!)\left(1+\frac{\lambda^2}{n}\right)\left(1-\frac{(x-\lambda)^2}{2n}\right),$$

from which he infers that the Poisson approximation will be good if λ and $x-\lambda$ are small relative to $\sqrt{n}$. He also determines λ as the modal value of the Poisson (λ) distribution; since the modal value is $[\lambda]$, he clearly has in mind λ an integer. He next shows that if X has such a distribution, that for large λ, $\Delta = X - \lambda$ has the approximate density

$$\frac{1}{\sqrt{2\pi\lambda}} \exp\{-\Delta^2/(2\lambda)\}\{1-(\Delta/2\lambda)\}$$

so that, for good approximation, Δ small relative to λ is required.

His introduction to the second paper shows his remarkable physical perception of the Poisson process, as well as conveying something of his statistical uncertainty. We give it in free translation from the German:

> 'Suppose objects are distributed randomly in some manner in space, or specific events randomly in time, or specific characteristics randomly within a set of discrete things; and it is required to determine the mean frequency of these objects or events or characteristics through counting in a known volume or time interval or subset. Then the result of a single counting (x) will be more or less than the mean value (λ) for the unit under investigation, which will result from repeated counting. We wish to determine the probability that ... in a single case the difference $(x-\lambda)$ lies outside specific limits absolutely or percentagewise.'

3. De Moivre and the concept of a martingale

Consider a Markov chain $\{X_n\}$, $n \geq 0$, on the states $\{0, 1, 2, \ldots, N\}$ where the states 0 and N are absorbing and passage from each of the states $\{1, 2, \ldots, N-1\}$ to 0 or N may occur in some finite number of steps with positive probability. Then it is well known that absorption into state 0 or N occurs with probability 1, and if $\{p_{ij}^{(k)}\}_{i,j=0}^{N}$ is the matrix of k-step transition probabilities for the chain, $p_{ij}^{(k)} \to 0$ as $k \to \infty$, $i, j = 1, \ldots, N-1$, and if $\gamma_{i,0}$ and $\gamma_{i,N}$ are the probabilities of ultimate absorption into 0 and N respectively $(\gamma_{i,0} + \gamma_{i,N} = 1)$ from $i = 1, \ldots, N-1$, $p_{i0}^{(k)} \to \gamma_{i,0}$ and $p_{iN}^{(k)} \to \gamma_{i,N}$. A particular well-known instance of such a Markov chain is the simple random walk with absorbing barriers, or the gambler's ruin problem, where $p_{i,i-1}^{(1)} = 1 - \pi$, $p_{i,i+1}^{(1)} = \pi$, $i = 1, \ldots, N-1$, $0 < \pi < 1$, X_n denoting a gambler's fortune.

In regard to one of the fundamental problems associated with such chains, that of determination of $\gamma_{i,0}$ or, equivalently, $\gamma_{i,N}$, in terms of the parameters involved in the specification of the process, the notion of a martingale has sometimes been used implicitly.

To take a specific instance, in the random walk example mentioned, Moran (1959–1960) notes that if $x \neq 0$ is an arbitrary number

$$\begin{aligned} \mathscr{E}(x^{X_{k+1}} | X_k) &= x^{X_k} + (\pi x + (1-\pi)x^{-1} - 1)x^{X_k}, \qquad & X_k \neq 0, N \\ &= x^{X_k} \qquad\qquad , & X_k = 0, N \end{aligned}$$

and hence if we take $x = (1-\pi)/\pi$, $\mathscr{E}(x^{X_{k+1}} | X_k) = x^{X_k}$. (Thus we see that, with this choice of x, $\{x^{X_k}\}$ is a martingale.) From the fact that then

$$x^i = \mathscr{E}(x^{X_k} | X_0 = i) = p_{i0}^{(k)} + \sum_{j=1}^{N-1} p_{ij}^{(k)} x^j + p_{i,N}^{(k)} x^N$$

letting $k \to \infty$ and using the fact $\gamma_{i,0} + \gamma_{i,N} = 1$, we obtain

$$\gamma_{i,N} = \frac{1 - x^i}{1 - x^N}, \qquad \text{where } x = (1 - \pi)/\pi.$$

It is remarkable that as early as 1711 (see Thatcher (1957)) De Moivre devised an ingenious argument for determining $\gamma_{i,0}$ and $\gamma_{i,N}$ in the same ('gambler's ruin') example, which is based in essence on a similar martingale device. He follows the fluctuation of

$$Y_n = \sum_{j=1}^{X_n} \{(1 - \pi)/\pi\}^j, \qquad n \geq 0$$

($\sum_{j=1}^{0} = 0$), and notices that

$$\mathscr{E}(Y_{n+1} - Y_n | X_n) = \{(1 - \pi)/\pi\}^{X_n + 1}\pi - \{(1 - \pi)/\pi\}^{X_n}\{1 - \pi\} = 0.$$

(Here we see that $\{Y_n, \mathscr{F}_n\}$ is a martingale, where $\mathscr{F}_n$ is the σ-field generated by $\{X_0, \ldots, X_n\}$.) It follows in particular that

$$\mathscr{E}\{Y_n\} = \mathscr{E}\{Y_0\} = \sum_{j=1}^{i} \{(1 - \pi)/\pi\}^j$$

if $X_0 = i$. Letting $n \to \infty$ we obtain similarly to the previous argument

$$\gamma_{i,N} \left\{ \sum_{j=1}^{N} \{(1 - \pi)/\pi\}^j \right\} = \sum_{j=1}^{i} \{(1 - \pi)/\pi\}^j$$

which is precisely De Moivre's reasoning.

Note added in proof. Mention of Abbe (1878) is made in Lancaster, H. O. (1950) Statistical control in haematology. *J. Hygiene* **48**, 402–417.

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