Chambers and the Coxeter complex

References:
- M. Ronan, 1989 'Lectures on Buildings'
- A. Thomas, 2018 'Geometry and Topological aspects of Coxeter groups and Buildings'

Plan for Today:

§ 1. Chamber systems.

§ 2. Coxeter groups and complexes.

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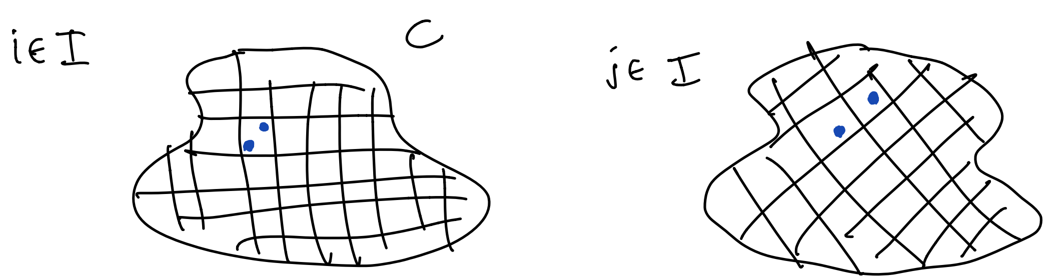
§ 1. Chamber systems

Let I be a set. A chamber system over I is a set C with an equivalence relation $\sim_i$ on it for each $i \in I$.

- Elements of C are called chambers.
- Two chambers x and y are called i-adjacent if $x \sim_i y$.



Example ① Let us take G a group, $B \subset G$ subgroup and I a set. Suppose for every $i \in I$ there is a subgroup P_i such that

$$B \subset P_i \subset G.$$

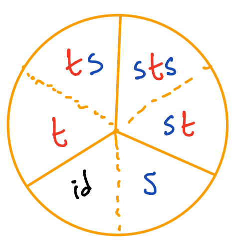
We can define a chamber system on $G/B := \{ gB \mid g \in G \}$ with the i-adjacency given by:

$$gB \sim_i hB \text{ if } gP_i = hP_i$$

Then G/B is a chamber system over I

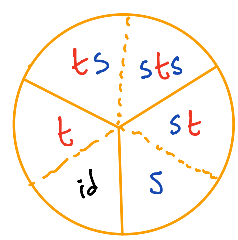
(2) As in (1) but $B = 1$ and $G = \langle s \in S \mid (st)^{m_{st}} = id, s^2 = id, \forall s,t \in S \rangle$. For $s \in S$, we define $P_s = \langle s \rangle$.

(3) As in (2) but $G = \langle s,t \mid (st)^3 = id, s^2 = t^2 = id \rangle$.
Here $S = \{s,t\}$



s-adjacency

$g \sim_s h$ if
$g\langle s \rangle = h \langle s \rangle$
if $g = h$ or
$g = hs$.



t-adjacency

A gallery is a finite sequence of chambers $(c_0, c_1, \dots, c_k)$ such that $\forall j$, $c_j \neq c_{j-1}$, $c_j \sim_{i_j} c_{j-1}$ for some $i_j \in I$.
The type of such gallery is $(i_1, i_2, \dots, i_k)$.
Let $J \subset I$, if $i_j \in J$ $\forall j$, we say the gallery is a J-gallery or gallery of type J.
We say two chambers x, y, are J-connected if there is a J-gallery $(x = c_0, c_1, \dots, c_k = y)$.

The J-connected components (maximal J-connected sets) are called J-residues or residues of type J.
The rank of a J-residue is $\mathrm{card}(J)$.

Remark/Definition:

- The residues of rank 0 are the same as the chambers.
- The " " " 1 are called panels.

e.g. In ③, $(t, ts, tst, ts, tst, st)$ is a gallery of type (s, t, t, t, s)

A morphism $\phi : (C, I) \longrightarrow (D, I)$ of two chamber systems over I is a map $C \longrightarrow D$ such that

$$\forall i \in I \quad x \sim_i y \Longrightarrow \phi(x) \sim_i \phi(y).$$

e.g. In ①. $G \curvearrowright G/B$ by left multiplication.
The map $\varphi_z : G/B \longrightarrow G/B$ is an automorphism
$\qquad gB \longmapsto zgB$ of $(G/B, I)$.

The geometric realisation Fix (C, I) a chamber system.
For I finite $\qquad (C, I) \rightsquigarrow$ geometric object.

Definition ① For R and S residues of type J and K respectively. Say S is a face of R if:

- $S \supset R$.
- $K \supset J$.

(2) Say cotype J for type $I - J$.

<u>Lemma:</u> Let R a residue of cotype J.

(i) For each $K \subset J$ there is a unique face of R of cotype K.

(ii) If S_1 and S_2 are two faces of R of cotypes K_1 and K_2, then S_1 and S_2 share the same face of cotype $K_1 \cap K_2$.

<u>Definition</u> An n-simplex is the convex portion of $n+1$ vertices in $\mathbb{R}^n$ in generic position. Each subset of those vertices spans a face.

Let (C, I) be a chamber system.

<u>Vertices</u>: One for each corank 1 residue.

<u>Edges:</u> One edge E for each residue R of corank 2, such that the faces of E are identified with the vertices constructed before associated to the faces of R.

Inductively: for a residue of cotype $\{i_1, \dots, i_r\}$ we construct a $(r-1)$-simplex σ s.t. their faces correspond to the faces of the residue.

This construction is called the geometric realisation of (C, I)

Example $C = \{x, y\}$, $I = \{1, 2, 3\}$ $x \sim_i y$ for $i \in I$

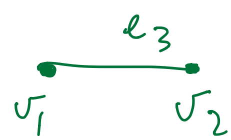
Vertices: cotype $1 \rightsquigarrow$ type $\{2,3\}$, $\{x,y\} \longrightarrow v_1$

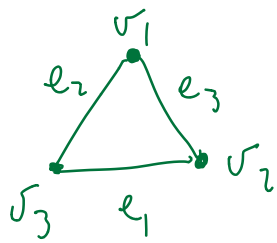
2 " $\{1,3\}$ $\{x,y\} \longrightarrow v_2$

3 " $\{1,2\}$ $\{x,y\} \longrightarrow v_3$

Edges: cotype $\{1,2\} \rightsquigarrow$ type $3 \rightsquigarrow \{x,y\} \rightsquigarrow e_3$

$$\{x,y\} \;||\; \{x,y\}.$$



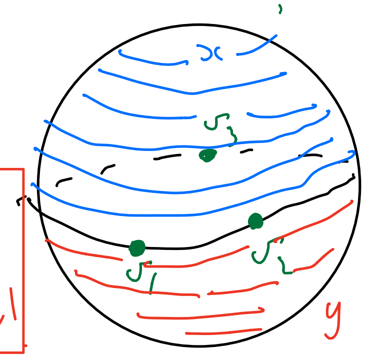


2-simplices corank 3 = type ϕ = chambers!

$$\{y\} \subset \{x, y\} \supset \{x\}$$

So everything above is a face!

This is not a simplicial complex!



§2. Coxeter groups and complexes

S set.

Let $M=\{m_{st}\}$ $\quad m_{st}\in\mathbb{Z}\cup\{\infty\}$ s.t. $s,t\in S$,

$$m_{s,t}\geq 2 \quad \text{for } s\neq t$$

$$m_{s,s}=1 \quad \forall s\in S.$$

The Coxeter group of type M is

$$W=\langle s\in S \mid s^2=id,\ (st)^{m_{st}}=id,\ \forall s,t\in S\rangle$$

W is a chamber system over S. The s-adjacency is given by

$$w\sim_s ws.$$

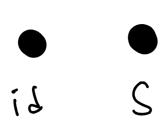
The Coxeter complex associated to (W,S) is the geometric realisation of this chamber system.

Examples: ① A_1: $W=\langle s\mid s^2=id\rangle=\{id,s\}$.

$S=\{s\}$.

Vertices: cotype $s\longrightarrow$ type $\phi=\{id,s\}$

Edges: cotype $\{s,t\}$? No edges.



$\cong S^0$.

② A_2, $W = \langle s,t \mid (st)^3 = id,\ s^2 = t^2 = id \rangle$

$S = \{s,t\}$.

<u>Vertices:</u>

cotype s	$\{id, t\} \leftarrow v_1$	cotype t	$\{id, s\} \leftarrow w_1$
$\Updownarrow$	$\{s, st\} \leftarrow v_2$	$\Updownarrow$	$\{t, ts\} \leftarrow w_2$
type t	$\{ts, tst\} \leftarrow v_3$	type s	$\{st, sts\} \leftarrow w_3$

<u>Edges:</u> corank 2 = type ϕ

$id \xrightarrow{\text{faces}} v_1$ & w_1

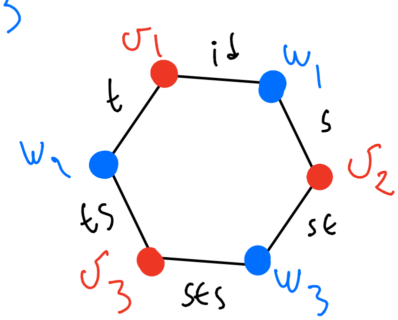
$s \longrightarrow v_2$ & w_1

$t \longrightarrow v_1$: w_2

$st \longrightarrow v_2$ w_3

$ts \longrightarrow v_3$ w_2

$sts \longrightarrow v_3$ w_3



Thank you!