QUESTION SHEET: CATEGORY THEORY WEEK 2

1. Exact functors

(1) Show that if a functor preserves short exact sequences, then it preserves any exact sequence.
(2) Show that the functor $\mathrm{Hom}_{\mathcal{C}}(-, Y) : \mathcal{C}^{op} \to Sets$ is left exact. Find a counterexample to show that it is not in general right exact.
(3) Show that the functor $X \otimes_R - : R\,\mathsf{Mod} \to R\,\mathsf{Mod}$ is right exact.

2. Hom Functors

(1) Let R be a commutative ring and let M, N be left R-modules. Show that the set $\mathrm{Hom}_R(M, N)$ is made into a left R-module by the following structure:
For an R-module homomorphism $f : M \to N$, $r \in R$, define

$$(r \cdot f)(m) = f(r \cdot m)$$

for all $m \in M$. Does this work when R is not commutative?
NB. By this excercise we have functors

$$\mathrm{Hom}_R(M, -) : R\,\mathsf{Mod} \to R\,\mathsf{Mod}$$
$$\mathrm{Hom}_R(-, N) : R\,\mathsf{Mod}^{op} \to R\,\mathsf{Mod}$$

whenever R is commutative.
(2) Let $[\mathcal{C}, Sets]$ be the category of functors $\mathcal{C} \to Sets$. In class we showed that there is a fully faithful functor $\mathcal{C} \to [\mathcal{C}^{op}, Sets]$ mapping an object $A \in \mathcal{C}$ to $\mathrm{Hom}_{\mathcal{C}}(-, A) : \mathcal{C}^{op} \to Sets$. Adapt the proof of this to show *Yoneda Lemma*:
Let $F : \mathcal{C}^{op} \to Sets$ and $X \in \mathcal{C}$. Show that there is a bijection

$$\mathrm{Hom}_{[\mathcal{C}^{op}, Sets]}(\mathrm{Hom}(-, X), FX) \cong FX$$

that is functorial in X.
(3) *Hard*: We say that an object of $[\mathcal{C}^{op}, Sets]$ is representable if it is isomorphic to a functor $\mathrm{Hom}_{\mathcal{C}}(-, X) : \mathcal{C}^{op} \to Sets$. Show that every object in $[\mathcal{C}^{op}, Sets]$ is a colimit of representable functors.

3. Adjoint functors

Given functors $F : \mathcal{C} \to \mathcal{D}$, and $G : \mathcal{D} \to \mathcal{C}$, say that (F, G) are an adjoint pair if the following equivalent conditions hold

- There is a natural transformation of functors $\eta : \mathrm{id}_{\mathcal{C}} \to GF$ (called a unit) such that for all morphisms $g : Y \to GZ$ in $\mathcal{C}$ there is a unique morphism $\bar{g} : FY \to Z$ in $\mathcal{D}$ such that

$$\begin{array}{ccc} GFY & \xrightarrow{G\bar{g}} & GZ \\ {\scriptstyle \eta_Y}\uparrow & \nearrow{\scriptstyle g} & \\ Y & & \end{array}$$

commutes.
- For any $X \in \mathcal{C}$, $Y \in \mathcal{D}$ there is an isomorphism $\mathrm{Hom}_{\mathcal{D}}(FX, Y) \cong \mathrm{Hom}_{\mathcal{C}}(X, GY)$ that is natural in both variables i.e. there is a natural isomorphism of bifunctors $\mathrm{Hom}_{\mathcal{D}}(F(-), -) \to \mathrm{Hom}_{\mathcal{C}}(-, G(-))$.

Questions:

(1) Show carefully that the above two definitions of adjoint functors are equivalent. Can you think of a third equivalent definition of an adjoint pair (F, G) involving a natural transformation $\epsilon : FG \to \mathrm{id}_{\mathcal{D}}$?

(2) Consider the functor: $F : Sets \to Vec_k$ mapping a set to the free vector space formally spanned by that set. Use the first definition of adjoint functors to show that F is left adjoint to the forgetful functor.

(3) Let R be a commutative ring and M an R-module. Show that $\mathrm{Hom}_R(M, -) : R\,\mathsf{Mod} \to R\,\mathsf{Mod}$ is left adjoint to $- \otimes_R M : R\,\mathsf{Mod} \to R\,\mathsf{Mod}$.

(4) Say that an object $P \in \mathcal{C}$ is projective if $\mathrm{Hom}_{\mathcal{C}}(P, -)$ is an exact functor. Use the previous excersise to show that if P, Q are projective R-modules then $P \otimes_R Q$ and $\mathrm{Hom}_R(P, Q)$ are projective R-modules. You may need that $\mathrm{Hom}(V, W) \cong V^* \otimes W$.

(5) Let $A \to B$ be a morphism of rings. This induces a functor $\mathsf{R}es : B\,\mathsf{Mod} \to A\,\mathsf{Mod}$ in the obvious way. Show that $\mathsf{R}es$ has left adjoint $\mathsf{Ind} = B \otimes_A - : A\,\mathsf{Mod} \to B\,\mathsf{Mod}$.

(6) Show that if $(F : \mathcal{C} \to \mathcal{D}, G : \mathcal{D} \to \mathcal{C})$ and $(H : \mathcal{D} \to \mathcal{E}, I : \mathcal{E} \to \mathcal{D})$ are adjoint pairs then (HF, GI) is an adjoint pair.

(7) Let A be a ring. In class we defined the functor $\mathrm{Sym}_A : A\,\mathsf{Mod} \to A\ \mathsf{CAlg}$ and showed that it is left adjoint to the forgetful functor $A\ \mathsf{CAlg} \to A\,\mathsf{Mod}$. Let $A \to B$ be a morphism of rings and M an A-module. Show that there is an isomorphism of B-algebras

$$\mathrm{Sym}_A(M) \otimes_A B \cong \mathrm{Sym}_B(M \otimes_A B)$$

which is functorial in M and is compatible with the grading. i.e. *Sym commutes with extension of scalars.*

(8) Show that if $G : \mathcal{D} \to \mathcal{C}$ has a left adjoint then G preserves products, pullbacks, and kernels. State and prove the dual of this statement. Note the following corollaries of this exercise:
- *Left adjoints are right exact.*
- *Right adjoints are left exact.*

(9) Use the previous exercise to show that there is an isomorphism of graded A-algebras

$$\mathrm{Sym}_A(M \oplus M') \cong \mathrm{Sym}_A(M) \otimes_A \mathrm{Sym}_A(M')$$

that is functorial in both M and M'.