ALGEBRAIC TOPOLOGY NOTES, PART I: HOMOLOGY

JONATHAN A. HILLMAN

1. Introduction

Algebraic topology advanced more rapidly than any other branch of mathematics during the twentieth century. Its influence on other branches, such as algebra, algebraic geometry, analysis, differential geometry and number theory has been enormous.

The typical problems of topology such as whether $\mathbb{R}^m$ is homeomorphic to $\mathbb{R}^n$ or whether the projective plane can be embedded in $\mathbb{R}^3$ or whether we can choose a continuous branch of the complex logarithm on the whole of $\mathbb{C}\backslash\{0\}$ may all be interpreted as asking whether there is a suitable continuous map. The goal of Algebraic Topology is to construct invariants by means of which such problems may be translated into algebraic terms. The homotopy groups $\pi_n(X)$ and homology groups $H_n(X)$ of a space X are two important families of such invariants. The homotopy groups are easy to define but in general are hard to compute; the converse holds for the homology groups.

In Part I of these notes we consider homology, beginning with simplicial homology theory. Then we define singular homology theory, and develop the properties which are summarized in the Eilenberg-Steenrod axioms. (These give an axiomatic characterization of homology for reasonable spaces.) We then apply homology to various examples, and conclude with two or three lectures on cohomology and differential forms on open subsets of $\mathbb{R}^n$. Although we shall assume no prior knowledge of Category Theory, we shall introduce and use categorical terminology where appropriate. (Indeed Category Theory was largely founded by algebraic topologists.)

Part II of these notes is an introduction to the fundamental group and combinatorial group theory.

We do not consider some topics often met in a first course on Algebraic Topology, such as reduced homology, or the Mayer-Vietoris Theorem, which is a very useful consequence of the Excision property. Other topics not considered here, but which are central to the further study and application of algebraic topology include the Jordan-Brouwer separation theorems, orientation for manifolds, cohomology and Poincaré duality.

References

Algebraic Topology: A First Course by M.J.Greenberg (2nd edition: and J.Harper), Benjamin/Cummings (1981).

Algebraic Topology, by A. Hatcher, Cambridge University Press (2002). Also available through the WWW ("www.math.cornell.edu/∼hatcher").

Further reading:

Lectures on Algebraic Topology by A. Dold, Springer-Verlag (1972).

A Concise Course in Algebraic Topology, by J.P.May, Chicago Lectures in Mathematics, Chicago UP (1999).

Algebraic Topology, by E.H.Spanier, McGraw-Hill (1966).

2. SOME SPACES

The most interesting spaces for geometrically minded mathematicians are manifolds, cell-complexes and polyhedra.

Notation.

$X \cong Y$ means X is homeomorphic to Y, if X, Y are spaces.

$\mathbb{R}^n_+ = \{(x_1, \dots, x_n) \in \mathbb{R}^n \mid x_n \geq 0\}$. $\partial\mathbb{R}^n_+ = \{(x_1, \dots, x_n) \in \mathbb{R}^n \mid x_n = 0\}$.

$D^n = \{(x_1, \dots, x_n) \in \mathbb{R}^n \mid \Sigma x_i^2 \leq 1\}$.

$S^{n-1} = \partial D^n = \{(x_1, \dots, x_n) \in \mathbb{R}^n \mid \Sigma x_i^2 = 1\}$.

$D^n_+ = \{(x_0, \dots x_n) \in S^n \mid x_n \geq 0\}$ and $D^n_- = \{(x_0, \dots x_n) \in S^n \mid x_n \leq 0\}$.

$intD^n = D^n \setminus \partial D^n = D^n \setminus S^{n-1}$. $O = (0, \dots, 0)$.

Exercise 1. *(a) Show that* $D^n \setminus \{O\} \cong S^{n-1} \times (0, 1]$ *and* $R^n \setminus \{O\} \cong S^{n-1} \times \mathbb{R}$.
(b) Show that $D^n \cong [0, 1]^n$.

Definition. An *n-manifold* is a space M whose topology arises from a metric and such that for all $m \in M$ there is an open neighbourhood U and a homeomorphism $h : U \to h(U)$ onto an open subset of $\mathbb{R}^n_+$. The *boundary* ∂M is the set of points m for which there is such a homeomorphism h with $h(m) \in \partial\mathbb{R}^n_+$.

We shall show later that the *dimension* n is well-defined, and that $\partial(\mathbb{R}^n_+) = \partial\mathbb{R}^n_+$, so ∂M is an $(n-1)$-manifold and $\partial\partial M = \emptyset$. (See Exercise 17 below.)

FACT. The metric condition is equivalent to requiring that M be Hausdorff (T_2) and that each connected component of M have a countable base of open sets. Open subspaces of $\mathbb{R}^n_+$ clearly satisfy these conditions, but there are bizarre examples which demonstrate that these conditions are not locally determined.

Examples. ($n = 2$): disc, sphere, torus (T), annulus, Möbius band (Mb).

The projective plane $P^2(\mathbb{R}) = S^2/(x \sim -x) \cong Mb \cup D^2$. Let $D_+ = \{(x, y, z) \in S^2 \mid z \geq \frac{1}{2}\}$, $D_- = \{(x, y, z) \in S^2 \mid z \geq \frac{1}{2}\}$ and $E = \{(x, y, z) \in S^2 \mid |z| \leq \frac{1}{2}\}$. Then $S^2 = D_+ \cup E \cup D_-$. Since the antipodal map interchanges D_+ and D_- and identifying antipodal points of E gives a Möbius band we see that $P^2(\mathbb{R}) \cong Mb \cup D^2$ is the union of a Möbius band with D^2.

All surfaces without boundary are locally homeomorphic to each other. We need a *global invariant* to distinguish them. Homology provides such invariants. (It is not so successful in higher dimensions.)

3. PROJECTIVE SPACES

Let $F = \mathbb{R}$, $\mathbb{C}$ or $\mathbb{H}$ (the quaternion algebra). Then F is a skew field and has finite dimension as a real vector space. Let $d = dim_{\mathbb{R}}F$ (= 1, 2 or 4).

Given a point $X = (x_0, \dots, x_n) \in F^{n+1} \setminus \{O\}$, let $[x_0 : \cdots : x_n]$ be the line through O and X in F^{n+1}. Let $P^n(F)$ (or FP^n) be the set of all such lines through O in F^{n+1}. Two nonzero points determine the same line if and only if they are proportional, i.e., $[x_0 : \cdots : x_n] = [y_0 : \cdots : y_n]$ if and only if there is a $\lambda \in F^\times = F \setminus \{O\}$ such that $y_i = \lambda x_i$ for $0 \leq i \leq n$. Hence $P^n(F) = (F^{n+1} \setminus \{O\})/F^\times$.

Each line through O in $F^{n+1} \cong \mathbb{R}^{(n+1)d}$ passes through the unit sphere $S^{(n+1)d-1}$, and two points on the unit sphere determine the same line if and only if one is a multiple of the other by an element of S^{d-1} ($= \{\pm 1\}$, S^1 or S^3), the subgroup of elements of $F^\times$ of absolute value 1. Thus there is a canonical surjection from $S^{(n+1)d-1}$ to $P^n(F)$, and $P^n(F) = S^{(n+1)d-1}/S^{d-1}$ is the orbit space of a group action. In particular, $P^n(\mathbb{R})$ is obtained from the n-sphere by identifying antipodal points.

The group $GL(n+1,F)$ acts transitively on the lines through O, and so $P^n(F)$ may be identified with the quotient of $GL(n+1,F)$ by the subgroup which maps the line $(x,0,\dots,0) \in F^{n+1}$ to itself.

Let $U_i = \{[x_0 : \dots : x_n] \in P^n(F) \mid x_i \neq 0\}$, for each $0 \leq i \leq n$. Then $P^n(F) = \cup_{i=0}^{i=n} U_i$. There are bijections $\phi_i : U_i \to F^n$, given by

$$\phi_i([x_0 : \dots : x_n]) = (x_0/x_i, \dots, \widehat{x_i/x_i}, \dots, x_n/x_i), \quad \text{for all} \quad [x_0 : \dots : x_n] \in U_i,$$

and with inverse $\phi_i^{-1}(y_1, \dots, y_n) = [y_1 : \dots : y_i : 1 : y_{i+1} : \dots : y_n]$. Moreover there is an obvious bijection from $P^{n-1}(F)$ to $P^n(F) \setminus U_i$.

As F^n has a natural metric topology, we may topologize $P^n(F)$ by declaring the subsets U_i to be open and the bijections ϕ_i to be homeomorphisms. We may identify D^{nd} with the unit ball in F^n. Then the map $h : D^{nd} \to P^n(F)$ given by $h(x_1, \dots, x_n) = [x_1 : \dots : x_n : 1 - |x|]$ is a continuous surjection. Hence $P^n(F)$ is compact. Moreover, h maps $intD^{nd} = D^{nd} \setminus S^{nd-1}$ homeomorphically onto U_n, while it maps $S^{nd-1} = \partial D^{nd}$ onto $P^{n-1}(F) = P^n(F) \setminus U_n$.

Exercise. *Show that $P^n(F)$ is Hausdorff and separable.*

A compact Hausdorff space is completely regular. If moreover it is separable then it is metrizable, by the Urysohn embedding theorem. (One can also define a metric on $P^n(F)$ directly.) Thus these projective spaces are manifolds.

Special cases. $P^1(F) = F \cup \{\infty\} = S^d$, where $\infty = [1 : 0]$, and points $[x : y] \in U_1$ are identified with the ratio $x/y \in F$. In particular, $P^1(\mathbb{C})$ is the extended complex plane. The map $h : S^3 \to S^2 = P^1(\mathbb{C})$ given by $h(u,v) = [u : v]$ for all $(u,v) \in \mathbb{C}^2$ such that $|u|^2 + |v|^2 = 1$ is known as the *Hopf fibration.*

Remark. The above construction of the *set* $P^n(F)$ works equally well for any skew field F, in particular for F a finite field!

4. CELL COMPLEXES

Definition. Let X be a space and $f : S^{n-1} \to X$ a map. Then $X \cup_f e^n = X \amalg D^n / (y \sim f(y), \forall y \in S^{n-1})$ is the space obtained by *adjoining an n-cell to X along f*. (The topology on $X \cup_f e^n$ is the finest such that the quotient function $q : X \amalg D^n \to X \cup_f e^n$ is continuous.)

We may identify X with a closed subset of $X \cup_f e^n$. The image of D^n is also closed (provided X is Hausdorff).

Note that since $S^{-1} = \partial D^0 = \emptyset$, we can adjoin 0-cells to the empty set.

Definition. A *finite cell-complex* is a space built from the empty set by successively adjoining finitely many cells of various dimensions.

Examples. $S^n = e^0 \cup_c e^n$. $S^1 \times S^1 = e^0 \cup e^1 \cup e^1 \cup_\gamma e^2$. (Consider the usual construction of the torus by identifying opposite sides of a rectangle.)

Let $h : D^{nd} \to P^n(F)$ be as defined in the previous section. If we identify $P^n(F) \setminus U_n$ with $P^{n-1}(F)$ then $h|_{S^{nd-1}}$ is the canonical map. Hence $P^n(F) = P^{n-1}(F) \cup e^{nd}$, and so we obtain a cell structure inductively. In particular, $P^n(\mathbb{R})$ has one cell in each dimension $\leq n$, $P^n(\mathbb{C})$ has one $2q$-cell for each $q \leq n$ and $P^n(\mathbb{H})$ has one $4q$-cell for each $q \leq n$.

We may also consider spaces constructed by adjoining infinitely many cells, but then the topology must be defined in a more complicated way. (Look up "*CW*-complex" in [Hatcher] or [Spanier].) We give just one example.

Identify D^n with $[0,1]^n$, and let $\iota_n(x) = (x, 0)$ for all $x \in D^n$ and $n \geq 1$. Then $\iota_n(D^n) \subset S^n = \partial D^{n+1} \subset D^{n+1}$. Let $S^\infty = \varinjlim(: S^n \to S^{n+1}) = \amalg S^n / \sim$ be the union of the spheres, with the "equatorial" identifications given by $s \sim \iota_{n+1}(s)$ for all $s \in S^n$. We give S^∞ the topology for which a subset $A \subset S^\infty$ is closed if and only if $A \cap S^n$ is closed for all n. Then S^∞ is a CW complex, with two cells in each dimension, represented by D^n and $\overline{S^n \setminus \iota_n(D^n)}$, for $n \geq 1$.

5. Polyhedra and simplicial homology

Definition. An *affine q-simplex* in $\mathbb{R}^N$ is the closed convex set determined by $q+1$ affinely independent points. The *standard* q-simplex Δ_q is determined by the standard basis vectors in $\mathbb{R}^{q+1}$. Thus

$$\Delta_q = \{(x_0, \dots, x_q) \in \mathbb{R}^{q+1} \mid \Sigma x_i = 1, x_j \geq 0 \ \forall j\}.$$

Clearly any affine q-simplex is homeomorphic to Δ_q, and so is a compact metric space.

Definition. The *faces* of the affine q-simplex determined by $\{P_0, \dots P_q\}$ are the affine simplices determined by subsets of this set.

Definition. A *polyhedron* in $\mathbb{R}^N$ is a subset P which is the union of finitely many affine simplices (of varying dimensions).

(We may assume that any two simplices meet along a common face, possibly empty.)

Definition. A *triangulation* of a space X is a homeomorphism $h : K \to X$ from some polyhedron K (in some $\mathbb{R}^N$).

In other words, a triangulation is a representation of X as a finite union of closed subsets, each homeomorphic to a simplex.

A polyhedron is a special case of a cell complex in which all the defining maps are one-to-one. Conversely, any such cell complex admits a triangulation as a polyhedron, but in general more simplices are needed than cells.

Example. $\partial\Delta_3 = \{(x_0, x_1, x_2, x_3) \in \mathbb{R}^4 \mid \Sigma x_i = 1, \Pi x_i = 0, x_j \geq 0 \ \forall j\}$ is a union of four 2-simplices, and is homeomorphic to the 2-sphere S^2, which has a cellular structure with just two cells. The torus $S^1 \times S^1$ has a cellular structure with just four cells. Its simplest triangulation requires seven vertices.

As each affine simplex is determined by its vertices, a polyhedron is determined by the set of all vertices together with the set of finite subsets corresponding to the simplices (and their faces). This is essentially combinatorial data, and there is a purely combinatorial notion of simplicial complex, which makes no mention of topology. (See [Spanier].) How can we use this to extract invariants of spaces which are insensitive to the triangulation used?

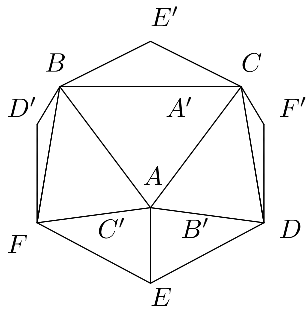
Euler observed that given any triangulation of the 2-sphere S^2 into V vertices, E "edges" and F "faces" (i.e., 0-, 1- and 2-simplices, respectively) then $V - E + F = 2$. This is easily proven by induction on V, once you show that any two triangulations have a common refinement. **Riemann** and **Betti** were lead to consider the "number of q-dimensional holes" in a space. It was only later that it was realized that these "Betti numbers" could be interpreted as the dimensions of certain vector spaces (or ranks of certain abelian groups), whose alternating sum gives an extension of Euler's invariant.

6. An example of simplicial homology

A q-dimensional hole in a polyhedron may perhaps be defined informally (in terms of what surrounds it) as a union of q-simplices which has no boundary - whose $(q-1)$-dimensional faces match up in pairs. We want to ignore such "q-cycles" which bound $(q+1)$-simplices. From "unions of q-simplices" to "formal sums" is not a great step. With the proper linear analogue of boundary - ∂ - we have $\partial\partial = 0$. We are lead to look at formal sums of q-simplices with $\partial = 0$ and to factor out $\partial((q+1)\text{-simplex})$. Thus we define $H_q(P) = \mathrm{Ker}(\partial)/\mathrm{Im}(\partial)$ as a measure of the q-dimensional holes.

Example. $S^2 = \partial\Delta_3$ may be triangulated as a tetrahedron, with vertices A, B, C, D, edges AB, AC, AD, BC, BD, CD and faces ABC, ABD, ACD, BCD. We take these as bases of vector spaces over a field $\mathbb{F}$, of dimensions 4, 6 and 4, respectively. The boundary maps are given by $\partial_0 V = 0$, for vertices V, $\partial_1 VW = W - V$, for edges VW, and $\partial_2 VWX = WX - VX + VW$, for faces VWX. These may be extended to linear maps betwen the vector spaces. We find that $\mathrm{Ker}(\partial_2)$ is 1-dimensional, generated by $BCD - ACD + ABD - ABC$, $\mathrm{Im}(\partial_2) = \mathrm{Ker}(\partial_1)$ is 3-dimensional and $\mathrm{Im}(\partial_1)$ is 3-dimensional, generated by the differences $B - A$, $C - A$ and $D - A$. Thus $H_2(S^2) \cong \mathbb{F}$, $H_1(S^2) = 0$ and $H_0(S^2) \cong \mathbb{F}$. (In fact, we could replace $\mathbb{F}$ by any ring R, provided we modify the terminology of "vector spaces" and "dimension".)

Exercise 2. *The triangulation of S^2 as an icosahedron (with 12 vertices, 30 edges and 20 faces) is is invariant under the antipodal map. Identifying opposite sides gives a triangulation of $P^2(\mathbb{R})$ with 6 vertices, 15 edges and 10 faces. Form the simplicial chain complex of this polyhedron, with coefficients in a commutative ring R. Compute the homology. Note in particular what happens if R is (a) a field of characteristic $\neq 2$, e.g., the real numbers $\mathbb{R}$; (b) the 2-element field $\mathbb{F}_2 = Z/2Z$; (c) the integers $\mathbb{Z}$.*



(The vertices A', B' and C' are "behind" the faces ABC, ADE and AEF, respectively, and the edges $A'B', A'C', A'D', A'E', A'F', B'C', B'D, B'E, B'F', C'D', C'E', C'F, D'E'$ and $E'F'$ are not shown.)

Many interesting spaces cannot be triangulated, and there are others which admit several essentially distinct triangulations with no common refinement. Thus it is not clear that simplicial homology provides a useful topological invariant. We shall have to modify our definition.

7. An algebraic interlude

The main algebraic prerequisite for Homology is linear algebra. We shall ultimately want to consider modules over a ring R, but you may assume that R is a field and the modules are vector spaces, for much of what follows.

The key notions that you should understand are homomorphism, kernel, image, submodule (or sub-vectorspace), quotient module (or quotient vectorspace) and cokernel.

Definition. A pair of module homomorphisms $j : M \to N$ and $q : N \to P$ is said to be *exact at* N if $\mathrm{Im}(j) = \mathrm{Ker}(q)$. These homomorphisms form a *short exact sequence* if the sequence $0 \to M \to N \to P \to 0$ is exact at each of M, N and P, i.e., if j is a monomorphism (one-to-one), q is an epimorphism (onto) and $\mathrm{Im}(j) = \mathrm{Ker}(q)$. Thus M is isomorphic to a submodule of N and $P \cong N/j(M)$. Note that if R is a field then $dim_R N = dim_R M + dim_R P$.

Definition. An R-*chain complex* C_* is a sequence of R-modules C_n and homomorphisms $\partial_n : C_n \to C_{n-1}$ such that $\partial_{n-1}\partial_n = 0$ for all n. The images and kernels of these *differentials* ∂_n define submodules $B_n = \mathrm{Im}(\partial_{n+1})$ and $Z_n = \mathrm{Ker}(\partial_n)$, called the *boundaries* and *cycles* in degree n, respectively. Clearly $B_n \leq Z_n \leq C_n$ (since $\partial\partial = 0$), and so we may define *homology* modules $H_n(C_*) = Z_n/B_n$.

In the cases of interest to us we shall usually have $C_n = 0$ for all $n < 0$.

Definition. A *homomorphism* α_* between two chain complexes C_* and D_* is a sequence of homomorphisms $\alpha_n : C_n \to D_n$ such that $\alpha_n\partial_{n+1} = \partial_{n+1}\alpha_{n+1}$, for all n. It follows easily that α_n maps boundaries to boundaries and cycles to cycles, and so induces a homomorphism $H_n(\alpha_*) : H_n(C_*) \to H_n(D_*)$. If $\gamma_* = \beta_*\alpha_*$ is the composite of two chain homomorphisms then $H_n(\gamma_*) = H_n(\beta_*)H_n(\alpha_*)$ for all n, while $H_n(id_{C_*}) = id_{H_n(C_*)}$. (In the language of categories, Homology is *functorial.*)

Exercise 3. *Let F be a field. Suppose that the following diagram of vector spaces and linear maps is commutative, with exact rows:*

$$\begin{array}{ccccccccc} A_1 & \longrightarrow & A_2 & \longrightarrow & A_3 & \longrightarrow & A_4 & \longrightarrow & A_5 \\ {\scriptstyle\alpha}\downarrow & & {\scriptstyle\beta}\downarrow & & {\scriptstyle\gamma}\downarrow & & {\scriptstyle\delta}\downarrow & & {\scriptstyle\epsilon}\downarrow \\ B_1 & \longrightarrow & B_2 & \longrightarrow & B_3 & \longrightarrow & B_4 & \longrightarrow & B_5 \end{array}$$

Let $p_i : A_i \to A_{i+1}$ and $q_j : B_j \to B_{j+1}$ be the horizontal maps. Show that (i) if α is onto and β and δ are 1-1 *then γ is* 1-1*; (ii) if β and δ are onto and ϵ is* 1-1 *then γ is onto. Identify clearly where each hypothesis is used.*

[Hint for (i): suppose that $a \in A_3$ and $\gamma(a) = 0$. Show $p_3(a) = 0$ hence $a = p_2(a')$ for some $a' \in A_2$. And so on ... Part (ii) is similar.] The argument applies without change to modules over a ring.

8. EXACT SEQUENCES OF COMPLEXES

A sequence of chain complexes $C_* \to D_* \to E_*$ is *exact* if each of the corresponding sequences of modules $C_n \to D_n \to E_n$ is exact at D_n, for all n. If $0 \to C_* \to D_* \to E_* \to 0$ is a short exact sequence of chain complexes then there are *connecting homomorphisms* from $H_n(E_*)$ to $H_{n-1}(C_*)$, for all n, giving rise to a *long exact sequence of homology*:

$$\ldots H_{n+1}(E_*) \xrightarrow{\ \delta\ } H_n(C_*) \to H_n(D_*) \to H_n(E_*) \xrightarrow{\ \delta\ } H_{n-1}(C_*) \to \ldots$$

Moreover the *connecting homomorphisms* δ are natural transformations: a morphism of short exact sequences of complexes gives rise to a commuting diagram with two parallel long exact sequences of homology.

In particular, if C_* is a *subcomplex* of D_* (i.e., if each homomorphism : $C_q \to D_q$ is a monomorphism) then the differentials induce differentials on the quotient modules D_q/C_q, and we obtain a short exact sequence

$$0 \to C_* \to D_* \to (D/C)_* \to 0.$$

The special case in which the complexes C_*, D_* and E_* are each trivial except in degrees 1 and 0 is known as the *Snake Lemma*, and then asserts that there is a six-term exact sequence

$$0 \to \mathrm{Ker}(\gamma_1) \to \mathrm{Ker}(\delta_1) \to \mathrm{Ker}(\epsilon_1) \to \mathrm{Cok}(\gamma_1) \to \mathrm{Cok}(\delta_1) \to \mathrm{Cok}(\epsilon_1) \to 0.$$

Conversely, the Snake Lemma can be used to establish the long exact sequence in general.

Exercise 4. *Show that the long exact sequence of homology determined by a short exact sequence of chain complexes is indeed exact.*

We shall later define the notion of *chain homotopy* between chain homomorphisms, and take the linear duals to obtain cochain complexes and cohomology.

9. Euler characteristic

Suppose now that R is a field and that all the vector spaces C_n are finite dimensional, and are 0 for all but finitely many values of n. The alternating sum of the dimensions $\chi(C_*) = \Sigma(-1)^n dim_R C_n$ is then a well-defined integer, called the *Euler characteristic* of C_*.

Exercise 5. *Show that*

$$\chi(C_*) = \Sigma(-1)^n dim_R H_n(C_*) = \Sigma(-1)^n (dim_F \mathrm{Ker}(\partial_n) - dim_F \mathrm{Im}(\partial_{n+1})).$$

(This is an elaboration of the formula $dim_R N = dim_R M + dim_R P$, if M is a subspace of N with quotient P.)

Exercise 6. *Show that any triangulation of the torus T requires at least 7 vertices.*

[Hint: Use the Euler characteristic, and note each edge is determined by its vertices, each edge is common to two faces and each face is triangular.]

Exercise 7. *Generalize (6) to the other closed surfaces.*

(For this you need to know that every such surface may be obtained from S^2 by replacing $g \geq 0$ disjoint closed discs with copies of the punctured torus $T_0 = T - intD^2$ in the orientable case, or by replacing $c \geq 1$ disjoint closed discs with copies of the Möbius band in the nonorientable case.)

Exercise. *Let G be a finite graph in which no pair of vertices is connected by more than one edge. Suppose that G is embedded in a surface S and let $\{F_i \mid i \in I\}$ be the set of components of $S \setminus G$. Show that $\chi(S) = V - E + \Sigma_{i \in I}\chi(F_i)$, where G has V vertices and E edges.*

10. SINGULAR HOMOLOGY

Let S be a set and R a ring. A function $f : S \to R$ is 0 *a.e.* if $f(s) = 0$ for all but finitely many $s \in S$. For each $s \in S$ let $e_s : S \to R$ be the function defined by $e_s(s) = 1$ and $e_s(t) = 0$ if $t \neq s$. Then e_s is 0 *a.e.* The *free R-module with basis S* is the set $R^{(S)} = \{f : S \to R \mid f \text{ is } 0 \text{ a.e.}\}$, with the obvious R-module structure. Every element in this module is uniquely expressible as a linear combination of the "basis" elements e_s. (In fact $f = \Sigma_{s \in S} f(s)e_s$.) It has the following "universal property": homomorphisms from $R^{(S)}$ to an R-module M correspond bijectively to functions from S to (the underlying set of) M.

Definition. A *singular q-simplex* in a space X is a map $\sigma : \Delta_q \to X$. The *module of singular q-chains on X with coefficients in R* is the free module $C_q(X; R)$ with basis the singular q-simplices in X. In other words, it is the set of finite formal sums $\Sigma r_\sigma \sigma$ where the coefficients r_σ are in the ring R and are 0 except for all but finitely many σs. Note that $C_q(X; R) = 0$ for all $q < 0$ and $C_0(X; R)$ is the free module with basis X. In general if $q \geq 0$ then $C_q(X; R)$ is usually huge - the basis is uncountable. We shall often simplify the notation to $C_q(X)$ if the coefficients are understood.

For each $0 \leq j \leq q$ let $F^j_q : \Delta_{q-1} \to \Delta_q$ be defined by $F^j_q(t_0, \ldots, t_{q-1}) = (t_0, \ldots, t_{j-1}, 0, t_j \ldots t_{q-1})$. Then the composite $\sigma^{(j)} = \sigma F^j_q$ is a singular $(q-1)$-simplex in X, called the j^{th} *face* of σ. We may now define the boundary operator $\partial_q : C_q(X) \to C_{q-1}(X)$ by $\partial(\sigma) = \Sigma(-1)^j \sigma^{(j)}$.

Exercise 8. *Verify that* $\partial_{q-1}\partial_q = 0$ *for all* $q \geq 2$.

Thus we obtain a chain complex, the singular chain complex of X, with coefficients R. The homology of this complex is the singular homology of X. Notation: $H_q(X; R)$ (or just $H_q(X)$). We shall see later that it agrees with the simplicial homology for polyhedra, but it is clearly defined for all spaces and depends only on the topology.

Exercise 9. *If* X *is a one-point space* $H_0(X; R) \cong R$ *and* $H_j(X; R) = 0$ *for all* $j > 0$.

Definition. A *path from P to Q in X* is a map $f : [a, b] \to X$ such that $f(a) = P$ and $f(b) = Q$. If any two points in X are the endpoints of such a path then X is *path-connected*; in general, every point is in some maximal path-connected subset of X, and X is a disjoint union of such *path components*.

Example. Any convex subset of $\mathbb{R}^N$ is path-connected. If X is path connected and $f : X \to Y$ is a map then $f(X)$ lies in some path component of Y.

A path in X determines a singular 1-simplex in X and so $H_0(X; R) \cong R$ if X is path-connected.

Let $\{X_\alpha \mid \alpha \in A\}$ be the set of path components of X, and for each path component X_α choose a point $P_\alpha \in X_\alpha$. Define a function $n_\alpha : C_0(X; R) \to R$ by $n_\alpha(x) = 1$ if $x \in X_\alpha$ and 0 otherwise. Let $m(\alpha) = [P_\alpha] \in H_0(X; R)$ for $\alpha \in A$ and $n([x]) = (n_\alpha(x)) \in R^{(A)}$ for $x \in X$. Then $mn = 1_{H_0(X;R)}$ and $nm = 1_{R^{(A)}}$.

Exercise 10. *Let* $X = S^0$ *be the* 0*-sphere, i.e., the* 2*-element discrete set. Show that if* R *is any ring* $H_0(X; R) \cong R^2$ *and* $H_q(X; R) = 0$ *if* $q > 0$.

In general, $H_q(X) \cong \bigoplus H_q(X_\alpha)$, where the sum is taken over all the path components X_α. As a consequence, we may often assume that our spaces are path connected.

Exercise 11. *Let X be the union of two disjoint open subsets Y and Z (i.e., $Y \cup Z = X$ and $Y \cap Z = \emptyset$). Show that $H_q(X) = H_q(Y) \oplus H_q(Z)$ for all q.*

Definition. Let R be a field. The q^{th} *Betti number* of X (with coefficients R) is $\beta_q(X;R) = dim_R H_q(X;R)$. If all the Betti numbers are finite, and are 0 for q large, then the *Euler characteristic* of X is $\chi(X;R) = \Sigma(-1)^q \beta_q(X;R)$.

The Betti numbers numbers depend on the coefficients R in general. We shall see that if X is a finite cell complex with c_q q-cells then $\chi(X;R) = \Sigma(-1)^q c_q$. In particular, this number is independent of the coefficient field, and moreover does not depend on how X is represented as a cell complex.

11. functoriality, relative homology, long exact sequence

Let $f : X \to Y$ be a map. If $\sigma : \Delta_q \to X$ is a singular q-simplex in X then $f\sigma$ is a singular q-simplex in Y. Thus f determines homomorphisms

$$C_q(f) : C_q(X) \to C_q(Y),$$

which are compatible with the differentials ($\partial_q^Y C_q(f) = C_{q-1}(f)\partial_q^X$) and so together give a chain homomorphism $C_*(f)$. It is easily verified that $C_*(id_X) = id_{C_*(X)}$ and $C_*(gf) = C_*(g)C_*(f)$, i.e., that the chain complex construction is functorial. Such chain homomorphisms induce homomorphisms

$$H_q(f) : H_q(X) \to H_q(Y),$$

which again are functorial.

Exercise 12. *Verify that H_q is a functor, i.e., that $H_q(1_X) = 1_{H_q(X)}$ and $H_q(fg) = H_q(f)H_q(g)$ for all q.*

We shall need to consider also *relative* homology, for pairs (X,A), where A is a subspace of X. The inclusion of A into X induces natural monomorphisms from $C_q(A)$ to $C_q(X)$, for all q. The quotient module $C_q(X,A) = C_q(X)/C_q(A)$ is the module of *relative q-chains.* Thus we have a short exact sequence of chain complexes

$$0 \to C_*(A) \to C_*(X) \to C_*(X,A) \to 0,$$

and a corresponding *long exact sequence of homology for the pair* (X,A).

We may describe the relative homology modules more explicitly as follows. Let $Z_q(X,A)$ be the module of q-chains c on X such that ∂c is in the image of $C_{q-1}(A)$, and let $B_q(X,A)$ be the submodule generated by $\partial C_{q+1}(X)$ and the image of $C_q(A)$. Then it follows easily from the standard isomorphism theorems of linear algebra that $H_q(X,A) \cong Z_q(X,A)/B_q(X,A)$.

We shall see that if A is a well-behaved subset of X then the relative homology depends largely on the difference $X \setminus A$; in this way we can hope to analyze the homology of a space by decomposing the space into simpler pieces.

We have shown that homology is functorial, the homology of a 1-point space is concentrated in degree 0, and that there is a long exact sequence of homology corresponding to any pair of spaces (X,Y). There remain two fundamental properties of homology that we must develop: "homotopy invariance" and "excision". Establishing these very important properties is rather delicate, and so we shall state them first, and defer the proofs, so that we can get on with computing the homology of a variety of spaces. (It can be shown that singular homology for cell-complexes may be characterized axiomatically by the properties listed in this paragraph.)

12. HOMOTOPY

If $f : X \to Y$ is a homeomorphism then (for all n) $H_n(f)$ is an isomorphism, since it has inverse $H_n(f^{-1})$. [As we have seen, $H_n(f) \circ H_n(f^{-1}) = H_n(f \circ f^{-1}) = H_n(id_Y) = id_{H_n(Y)}$, and similarly $H_n(f^{-1}) \circ H_n(f) = id_{H_n(X)}$.)

In fact much weaker conditions on f imply that the induced homomorphisms are isomorphisms. We shall see that homology is a rather coarse invariant, in that it cannot distinguish $\mathbb{R}^n$ from a point.

Definition. Two maps $f, g : X \to Y$ are *homotopic* if there is a map $F : X \times [0, 1] \to Y$ such that $f(x) = F(x, 0)$ and $g(x) = F(x, 1)$ for all $x \in X$. We shall write $f \sim g$.

Such a *homotopy* F determines a 1-parameter family of maps $F_t : X \to Y$ by $F_t(x) = F(x, t)$ for all $x \in X$, which we may think of as a path in the space of all maps from X to Y. We use the above formulation to avoid discussing the topology of spaces of maps.

Being homotopic is an equivalence relation: $f \sim f$; if $f \sim g$ then $g \sim f$; if $f \sim g$ and $g \sim h$ then $f \sim h$.

Let $[X; Y]$ be the set of homotopy classes of maps from X to Y. If $f \sim g : X \to Y$ and $h \sim k : Y \to Z$ then $hf \sim kg : X \to Z$. Since composition of homotopy classes is well-defined, we may define the homotopy category $((Hot))$, with objects topological spaces and $Hom_{((Hot))}(X, Y) = [X; Y]$.

Definition. A map $f : X \to Y$ is a *homotopy equivalence* if there is a map $h : Y \to X$ such that $hf \sim id_X$ and $fh \sim id_Y$. We then say that X and Y are homotopy equivalent, and write $X \simeq Y$. (We write $X \cong Y$ if X and Y are homeomorphic.)

Example. $\mathbb{R}^n$ is homotopy equivalent to a point. $\mathbb{R}^n \setminus \{O\} \simeq S^{n-1}$.

Homotopy equivalence is again an equivalence relation.

Exercise 13. *Show that if there are maps $g, h : Y \to X$ such that $gf \sim id_X$ and $fh \sim id_Y$ then $g \sim h$. In other words, if f has left and right inverses then they must be homotopic, and f is a homotopy equivalence.*

Definition. A space X is *contractible* if it is homotopy equivalent to a one point space.

Exercise 14. *Let $f, g : S^{q-1} \to A$ be homotopic maps. Show that $A \cup_f e^q$ and $A \cup_g e^q$ are homotopy equivalent.*

[Hint: we know very little about A, so any map $\phi : A \cup_f e^q \to A \cup_g e^q$ that we construct can only have value $\phi(a) = a$ for points $a \in A$. On the other hand, we may relate the homotopy parameter t to the radial coordinate in the k-disc D^k used in the construction of these spaces.]

Exercise 15. *Let $f : S^{k-1} \to X$ be a map and $g : X \to Y$ a homotopy equivalence. Show that $Y \cup_{gf} e^k$ is homotopy equivalent to $X \cup_f e^k$.*

[Hint: As in Exercise 14 we must use the given maps and implicitly given homotopies on X and Y. The only room for constructing new maps is in defining maps on the images of D^k.]

(The last two exercises are moderately difficult, but do not require any algebraic topology beyond the definition of homotopy.)

In particular, if X is path connected then $X \cup_f e^1 \simeq X \vee S^1$.

Claim (to be proven later:) if $f \sim g$ then $H_n(f) = H_n(g)$ for all n.

In particular, homotopy equivalent spaces have isomorphic homology. For if $hf \sim id_X$ and $fh \sim id_Y$ then $H_q(h)H_q(f) = H_q(hf) = H_q(id_X) = id_{H_q(X)} \dots$.

Hence $H_n(\mathbb{R}^m)$ is 0 for all $n > 0$ and all $m \geq 0$. (On the other hand we shall see that if we delete the origin then the punctured euclidean spaces $\mathbb{R}^m \setminus \{O\}$ all may be distinguished by their homology.) It can be shown that there is a near-converse: if X and Y are "simply-connected" cell-complexes (e.g. if they have no 1-cells) and $f : X \to Y$ induces isomorphisms on all homology groups then f is a homotopy equivalence. (Whitehead's Theorem: this is beyond the scope of this course.)

Example. The homology of $S^n = D^n_+ \cup D^n_-$. Let $N = (0, \dots 0, 1) \in S^n$ be the "north pole".

Claim (using excision; proven later): the inclusion of the pair $(D^n_+, D^n_+ \setminus \{N\})$ into $(S^n, S^n \setminus \{N\})$ induces isomorphisms on all homology groups.

Now D^n_+ and $S^n \setminus \{N\}$ are both contractible, while $D^n_+ \setminus \{N\} = D^n_+ \cap D^n_- \simeq S^{n-1}$. On applying the long exact sequences of homology for these spaces and making use of the isomorphisms coming from excision and homotopy equivalences we find that $H_q(S^n) \cong H_{q-1}(S^{n-1})$, provided $q \geq 2$. If $q = 1$ we get an exact sequence

$$0 \to H_1(S^n) \to H_0(S^{n-1}) \to H_0(D^n_+).$$

Since $S^0 = \{1, -1\}$ and S^n is path connected if $n > 0$ it follows that $H_1(S^n) = 0$ unless $n = 1$, in which case $H_1(S^1) \cong R$. Since $H_0(S^0) \cong R^2$ and $H_q(S^0) = 0$ for $q > 0$ we may conclude that $H_q(S^n) = 0$ unless $q = 0$ or n, while $H_0(S^n) \cong H_n(S^n) \cong R$ if $n > 0$.

Exercise 16. *The cone over a space W is the space $CW = W \times [0,1]/W \times \{0\}$ (i.e., identify all points in $W \times \{0\}$ to one point). Show that CW is contractible. Let CW_+ and CW_- be two copies of CW. The suspension of W is the space SW obtained by identifying the copies of $W \times \{1\}$ in CW_- and CW_+, i.e., $SW = CW_- \cup CW_+$, where $CW_- \cap CW_+ = W$. Assuming that the inclusion of (CW_-, W) into (SW, CW_+) induces isomorphisms on relative homology, prove that $H_q(SW) \cong H_{q-1}(W)$ if $q > 1$. What can you say about $H_0(SW)$, $H_1(SW)$?*

Exercise 17. *Let V be an open subset of $\mathbb{R}^n$, and $p \in V$. Show that $H_{n-1}(S^{n-1})$ is a direct summand of $H_{n-1}(V \setminus \{p\})$.*

Deduce that if $\mathbb{R}^m$ is homeomorphic to an open subset of $\mathbb{R}^n$ then $m = n$.

A variation on this argument shows that $\partial(\mathbb{R}^n_+) = \partial\mathbb{R}^n_+$, since points $p \in \partial\mathbb{R}^n_+$ have small open neighbourhoods V such that $V \setminus \{p\}$ is contractible.

Exercise. *Show that S^∞ is contractible.*

[Hint: We may also write $S^\infty = \varinjlim(: D^n \to D^{n+1})$ as in §4.]

One might suppose that S^∞ is the unit sphere in ℓ^2_∞, the (countably) infinite dimensional analogue of $\mathbb{C}^n$. However the "obvious " map is not a homeomorphism. Let $U = \{x \in \ell^2_\infty \mid ||x|| = 1\}$, and let U_0 be the subset of points with almost all coordinates 0. The natural map from S^∞ to U_0 is a continuous bijection. However the subset $C = \{\frac{n}{\sqrt{n^2+1}}(e_1 + \frac{1}{n}e_n) \mid n \geq 2\}$ is closed in S^∞, but is not closed in U_0, and so this bijection is not a homeomorphism. On the other hand U_0 is a proper subset of U, which is dense in U. Since the standard basis vectors form an infinite discrete subset of U_0, neither U_0 nor U is compact. Therefore no map with compact domain is surjective. It follows easily that $\pi_n(U) = 0$ for all n. Is U (or U_0) contractible? (Since S^∞ and U_0 are countable unions of nowhere dense subsets, they are not homeomorphic to U, by Baire's Theorem.)

13. HOMOTOPIC MAPS INDUCE THE SAME HOMOMORPHISM

Theorem 1. *Let f and g be homotopic maps from X to Y. Then $H_q(f) = H_q(g)$ for all q.*

Proof. We shall show that this follows by functoriality from a special case. Let $F : X \times [0,1] \to Y$ be a homotopy from $F_0 = f$ to $F_1 = g$, and let $i_t : X \to X \times [0,1]$ be the map defined by $i_t(x) = (x,t)$ for all $x \in X$ and $0 \leq t \leq 1$. Then $F_t = Fi_t$, so $H_q(f) = H_q(F)H_q(i_0)$ and $H_q(g) = H_q(F)H_q(i_1)$, for all q. Thus it shall suffice to show that $H_q(i_0) = H_q(i_1)$, for all q. In other words, we may assume that $Y = X \times [0,1]$ and $F = id_Y$ (so that $F_t = i_t$).

We shall define "prism" operators $P_q : C_q(X) \to C_{q+1}(X \times [0,1])$ such that $C_q(i_1) - C_q(i_0) = \partial_{q+1}^{X\times[0,1]} P_q + P_{q-1}\partial_q^X$. Thus if ζ is a singular q-cycle on X we have $C_q(i_1)(\zeta) - C_q(i_0)(\zeta) = \partial_{q+1}^{X\times[0,1]} P_q(\zeta)$, and so $H_q(i_1)([\zeta]) = H_q(i_0)([\zeta])$. In order to do this, we shall specialise further, to the case $X = \Delta_q$. Let $E_i = (0,\dots,0,1,0\dots)$ be the i^{th} vertex of Δ_q, and let $A_i = (E_i, 0)$ and $B_i = (E_i, 1)$, for $0 \leq i \leq q$. Then $\{A_i, B_i\}$ are the $2q+2$ vertices of $\Delta_q \times [0,1]$. Let $\pi_{q+1} = \Sigma_{i=0}^{i=q}(-1)^i (A_0, \dots, A_i, B_i, \dots, B_q)$, where $(A_0, \dots, A_i, B_i, \dots, B_q)$ is the singular $(q+1)$-simplex determined by the linear map which sends E_j to A_j if $j \leq i$ and to B_{j-1} if $j > i$. Thus π_{q+1} is the singular $(q+1)$-chain on $\Delta_q \times [0,1]$ corresponding to a particular triangulation of this "prism on Δ_q" as a union of affine $(q+1)$-simplices. Note that the boundary of this prism consists of the top and bottom copies of Δ_q and the union of the prisms on the $(q-1)$-dimensional faces of Δ_q.

Define $P_q : C_q(X) \to C_{q+1}(X \times [0,1])$ by $P_q(\sigma) = C_{q+1}(\sigma \times id_{[0,1]})(\pi_{q+1})$ for any singular q-simplex σ in X. We may then check that $\partial_{q+1}^{X\times[0,1]} P_q = C_q(i_1) - C_q(i_0) - P_{q-1}\partial_q^X$, and so these homomorphisms P_q are as required. □

Remark. The name "prism" is suggested by the shape of $\Delta_2 \times [0,1]$.

The maps $C_{q+1}(F)P_q$ determine a *chain homotopy* from $C_*(f)$ to $C_*(g)$.

Exercise. *Work through the details for $q = 2$.*

We shall need also a *relative* version of this result. Two maps $f, g : (X,A) \to (Y,B)$ are homotopic as maps of pairs if there is a homotopy F_t from f to g such that $F_t(A) \subseteq B$ for all $0 \leq t \leq 1$.

Exercise. Prove the following

Theorem 2. *Let $f, g : (X,A) \to (Y,B)$ be homotopic as map of pairs. Then $H_q(f) = H_q(g)$ for all q.*

(This is tedious to do properly, and I shall not attempt it in class.)

14. EXCISION

Let X be a space with a subspace A, and let $U \subseteq A$. Then there is an inclusion of pairs $(X \setminus U, A \setminus U) \to (X, A)$.

Claim: [Excision] *If the closure of U is contained in the interior of A then this inclusion induces isomorphisms on relative homology, $H_q(X\setminus U, A\setminus U; R) \cong H_q(X, A; R)$, for all q and any coefficients R.*

The hypotheses on U can be relaxed considerably, in conjunction with homotopy invariance. We shall defer the proof of this excision property for several lectures, and instead concentrate on using it. We have already sketched one use, to compute the homology of the spheres.

Examples. (1) The figure-eight $X = S^1 \vee S^1$. This is the one point union of two circles (e.g., the set $\{(x, y) \in \mathbb{R}^2 \mid (x^2 + y^2)^2 = 4x^2\}$). Let E, W be the midpoints of the two circles and let O be the point of intersection of the two circles. Let $A = X \setminus \{E, W\}$ and $U = \{O\}$. Then A is contractible, while $X \setminus U$ and $A \setminus U$ are the disjoint unions of two and four contractible pieces (arcs), respectively. It follows easily that $H_q(X) = 0$ if $q > 1$, $H_1(X) \cong \mathbb{Z}^2$ and $H_0(X) \cong \mathbb{Z}$.

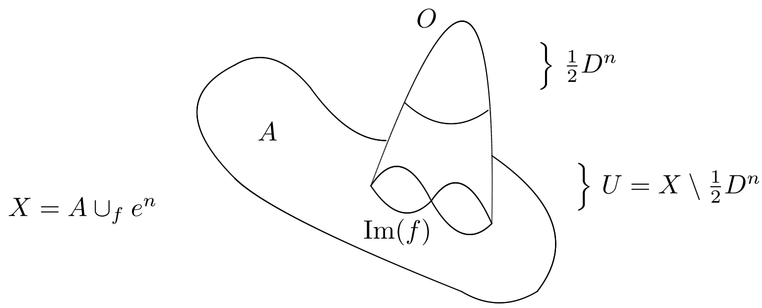
(2) Adjunction spaces. (See figure.) Let $X = A \cup_f e^n$ and let $B = X \setminus \{O\}$, where O is the centre of the n-cell adjoined to A. Let $U = X \setminus \frac{1}{2}D^n$. The natural inclusion $i : A \to B$ is a homotopy equivalence, with homotopy inverse the map $j : B \to A$ such that $j(a) = a$ for all $a \in A$ and $j(d) = f(d/||d||)$ for all $d \in B \setminus A$. (The maps $f_t : B \to B$ given by $f_t(a) = a$ for all $a \in A$, $f_t(d) = d/(1 - t + t||d||)$ if $d \in B \setminus A$ and $t < 1$ and $f_1(d) = j(d)$ define a homotopy from $f_0 = id_B$ to ij.) Hence the homomorphisms $H_q(X, A) \to H_q(X, B)$ are isomorphisms, by the "5-Lemma". (See Exercise 2.) The homomorphisms

$$H_q(\frac{1}{2}D^n, \frac{1}{2}D^n \setminus \{O\}) = H_q(X \setminus U, B \setminus U) \to H_q(X, B)$$

and

$$H_q(\frac{1}{2}D^n, \frac{1}{2}D^n \setminus \{O\}) \to H_q(D^n, D^n \setminus \{O\})$$

are isomorphisms, by excision, while $H_q(D^n, S^{n-1}) \cong H_q(D^n, D^n \setminus \{O\})$, by homotopy invariance.



We find that $H_q(A) \cong H_q(X)$ if $q \neq n - 1$ or n, while there is an exact sequence

$$0 \to H_n(A) \to H_n(X) \to H_n(X, A)(\cong R) \to H_{n-1}(A) \to H_{n-1}(X) \to 0.$$

The map $f : S^{n-1} \to A$ extends to a map of pairs $(D^n, S^{n-1}) \to (X, A)$, which induces a map between the long exact sequences of homology for these pairs. Assume for simplicity that $n > 1$. Then the connecting homomorphism from $H_n(D^n, S^{n-1})$ to $H_{n-1}(S^{n-1})$ is an isomorphism. As the homomorphism from $H_n(D^n, S^{n-1})$ to $H_n(X, A)$ is also an isomorphism (by excision), we see that the connecting homomorphism from $H_n(X, A)$ to $H_{n-1}(A)$ is the composite of $H_{n-1}(f) : H_{n-1}(S^{n-1}) \to H_{n-1}(A)$ with an isomorphism, and in particular has the same image as $H_{n-1}(f)$. In later examples we shall want to identify this homomorphism more explicitly.

Exercise 18. *Let X be a finite cell complex of dimension at most n. Show that $H_q(X;\mathbb{Z})$ is finitely generated for all q, and is 0 for all $q > n$.*

Exercise 19. *Let X be a finite cell complex of dimension at most n. Show that $H_n(X;\mathbb{Z})$ is torsion free.*

15. Euler characteristic, graphs and surfaces

Without knowing anything about the homomorphisms induced by attaching maps we may still obtain nontrivial results. Let X be a finite cell complex, with c_q q-cells, and let F be a field. Let $\beta_q(X;F) = dim_F H_q(X;F)$. Then these dimensions are finite and are 0 for q large, and $\Sigma(-1)^q c_q = \Sigma(-1)^q\beta_q(X;F)$. This number is the Euler characteristic of X, and the equation shows that it is independent of how X is represented as a cell-complex, and also independent of the coefficient field (although the individual Betti numbers $\beta_q(X;F)$ do depend on F).

The proof is by induction on Σc_q, the number of cells in X. The result is clearly true for $X = \emptyset$). Suppose that it holds for the finite cell-complex A, and that $X = A \cup_f e^n$. Then $\beta_q(X) = \beta_q(A)$ if $q \neq n-1$ or n, while

$$\beta_n(A;F) + 1 - \beta_{n-1}(A;F) = \beta_n(X;F) - \beta_{n-1}(X;F)$$

(from the five-term sequence on the previous page). Since $c_q(X) = c_q(A)$ unless $q = n$, while $c_n(X) = c_n(A) + 1$, the inductive step follows easily.

Exercise 20. *Let X and Y be finite cell complexes. Show that $X \times Y$ is a finite cell complex and $\chi(X \times Y) = \chi(X)\chi(Y)$.*

[Hint: as a set, a finite cell complex is the disjoint union of the "interiors" of its cells (where the interior of a q-cell e^q of X is the image of $intD^q$ in X). It follows immediately that $X \times Y$ is the disjoint union of subsets corresponding to $intD^{p+q} = intD^p \times intD^q$. The cells of a cell-complex are ordered, and are attached to the unions of preceding cells. Use a lexical ordering on the cells of the product. Since $\partial D^p \times D^q = S^{p-1} \times D^q \cup D^p \times S^{q-1}$ the attaching maps for the cells of $X \times Y$ may be described in terms of the attaching maps for the cells of X and Y.

Evaluate χ by counting the cells: do NOT attempt to compute the homology!]

Exercise 21. *Show that if $0 < m < n$ then $H_i(S^m \times S^n; R) \cong R$ if $i = 0, m, n$ or $m+n$. What is the homology of $S^n \times S^n$?*

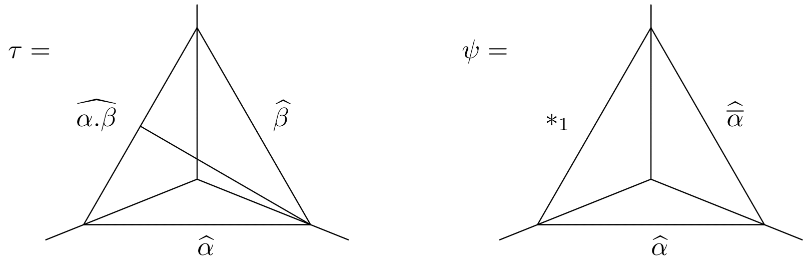
[Note that $S^p \times S^q = (S^p \times D^q_-) \cup (D^p_- \times S^q) \cup (D^p_+ \times D^q_+)$.]

16. Some explicit chains

Let $p : \Delta_1 \to [0,1]$ be the homeomorphism given by projection onto the Y-axis: $p(x,y) = y$ if $(x,y) \in \Delta_1$. A path $\gamma : [0,1] \to X$ determines a singular 1-simplex $\widehat{\gamma} = \gamma p$.

1. Let $X = I = [0,1]$, $\partial I = \{0,1\}$ and $\gamma = id_I$. Then $\widehat{id_I}$ is a relative 1-cycle on $(I, \partial I)$ and its homology class generates $H_1(I, \partial I) \cong \mathbb{Z}$. (Consider the long exact sequence of homology for the pair $(I, \partial I)$, and observe that $\delta[\widehat{id_I}] = [1] - [0]$ generates the kernel of the natural homomorphism from $H_0(\partial I) \cong \mathbb{Z}^2$ to $H_0(I) \cong \mathbb{Z}$.)

2. Let α, $\beta : [0,1] \to X$ be paths in X with $\alpha(1) = \beta(0)$ and define a path $\alpha.\beta$ by concatenation: $\alpha.\beta(t) = \alpha(2t)$ if $0 \leq t \leq \frac{1}{2}$ and $\alpha.\beta(t) = \beta(2t-1)$ if $\frac{1}{2} \leq t \leq 1$. Let $\tau : \Delta_2 \to X$ be the singular 2-simplex defined by $\tau(x,y,z) = \alpha(1-x+z)$ if $x \geq z$ and $\tau(x,y,z) = \beta(z-x)$ if $z \geq x$. Then $\partial\tau = \widehat{\beta} - \widehat{\alpha.\beta} + \widehat{\alpha}$.



3. Let $\bar{\alpha}(t) = \alpha(1-t)$ and let $\psi : \Delta_2 \to X$ be the singular 2-simplex defined by $\psi(x,y,z) = \alpha(y)$. For any $q \geq 0$ let $*_q$ be the constant singular q-simplex with value $\alpha(0)$. Then $\partial(\psi + *_2) = \widehat{\bar{\alpha}} - *_1 + \widehat{\alpha} + *_1 - *_1 + *_1 = \widehat{\bar{\alpha}} + \widehat{\alpha}$.

4. Let P be a polygon with n sides α_j, for $1 \leq j \leq n$ (numbered consecutively). Then $\sigma = \Sigma\widehat{\alpha_j}$ is a 1-cycle on $\partial P \cong S^1$, which generates $H_1(\partial P) \cong \mathbb{Z}$. For let $U = \partial P \setminus int\alpha_1$. Then $H_1(\partial P) \cong H_1(\partial P, U) \cong H_1(\alpha_1, \partial\alpha_1)$, and the images of σ and $\widehat{\alpha_1}$ in $H_1(\partial P, U)$ agree. (Here $\widehat{\alpha_1}$ is considered as a relative 1-cycle on $(\alpha_1, \partial\alpha_1) \cong (I, \partial I)$.)

5. Similarly, if $\epsilon(t) = e^{2\pi i t}$ for $0 \leq t \leq 1$ then $\widehat{\epsilon}$ generates $H_1(S^1)$. Moreover, if $\alpha_j(t) = e^{2\pi i(j-1+t)/n}$, for $1 \leq j \leq n$ then $[\widehat{\epsilon}] = [\Sigma\widehat{\alpha_j}]$ in $H_1(S^1)$.

Exercise. *Let P, Q be points in a path-connected space X, and let $A : [0,1]^2 \to X$ be a homotopy of paths α_t, where $\alpha_t(s) = A(s,t)$ for all $0 \leq s,t \leq 1$. Suppose that $\alpha_t(0) = P$ and $\alpha_t(1) = Q$ for all $0 \leq t \leq 1$. Show that $\widehat{\alpha_1} - \widehat{\alpha_0} = \partial\Theta$, where Θ is a singular 2-chain constructed using the homotopy A.*

[Divide the domain of the homotopy into two "prisms", as in the proof of Theorem 1, the homotopy invariance theorem. Note that $\widehat{\alpha_0}$ and $\widehat{\alpha_1}$ are relative cycles for the pair $(X, \{P,Q\})$, and that $[\widehat{\alpha_0}] = [\widehat{\alpha_1}]$ in $H_1(X, \{P,Q\})$, by Theorem 2.]

6. Let let $f_n : S^1 \to S^1$ be the n^{th} power map: $f_n(z) = z^n$ for all $z \in S^1$. Since f_0 is the constant map $H_1(f_0) = 0$. If $n \geq 1$ then $f_n\alpha_j = \epsilon$ for $1 \leq j \leq n$. Hence $H_1(f_n)([\widehat{\epsilon}]) = [\Sigma\widehat{f_n\alpha_j}] = n[\widehat{\epsilon}]$ in $H_1(S^1)$.

If $n < 0$ then $f_n\alpha_j = \bar{\epsilon}$ for $1 \leq j \leq |n|$. Hence $H_1(f_n)([\widehat{\epsilon}]) = |n|[\widehat{\bar{\epsilon}}] = n[\widehat{\epsilon}]$. Thus f_n induces multiplication by n on $H_1(S^n)$, for all n.

7. Generalization (not proven). Let P be a convex polyhedron in R^n, whose faces are triangulated as $(n-1)$-simplices. Then if these faces are consistently oriented their formal sum represents a generator of $H_{n-1}(\partial P) \cong H_{n-1}(S^{n-1}) \cong \mathbb{Z}$.

Exercise 22. *The fundamental theorem of algebra. Let $P(z) = z^n + a_{n-1}z^{n-1} + \ldots + a_0$ be a polynomial of degree $n \geq 1$ with complex coefficients. Show that if r is large enough $P_t(z) = (1-t)P(z) + tz^n$ has no zeroes on the circle $|z| = r$ for any $0 \leq t \leq 1$. Hence the maps : $z \to z^n$ and : $z \to P(rz)/|P(rz)|$ are homotopic as maps from S^1 to S^1. If P has no zeroes the latter map extends to a map from the unit disc D^2 to S^1. CONTRADICTION. Why?*

17. WEDGE, BOUQUETS, THE TORUS

Let X and Y be spaces with given basepoints x_0, y_0. The *one-point union* of X and Y is $X \vee Y = (X \amalg Y)/(x_0 = y_0)$. (Thus $S^1 \vee S^1$ is the figure eight.) The common image of $x_0 = y_0$ represents a natural basepoint for $X \vee Y$.

Assume that the base-points have contractible neighbourhoods in X and Y, respectively. If X and Y are path-connected then so is $X \vee Y$, and $H_q(X \vee Y) \cong H_q(X) \oplus H_q(Y)$ for all $q > 0$.

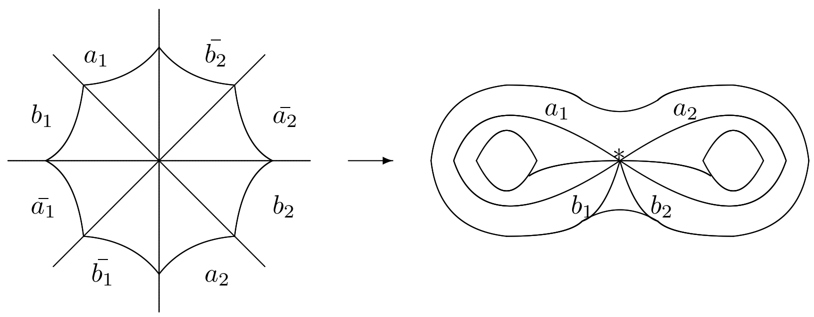
Example. $\vee^r S^1$ is defined inductively. It may also be obtained by adjoining r 1-cells to a point.

The torus $T = S^1 \times S^1$ may be constructed by adjoining a 2-cell to $S^1 \vee S^1$. More precisely, we may construct T by identifying opposite sides of the unit square R with vertices A, B, C, D. It is clearly path-connected, so $H_0(T; R) = R$, and $H_q(T; R) = 0$ if $q > 2$, since T has no cells of dimension > 2. The 1-cycle $z = AB.BC.CD.DA$ generates $H_1(\partial R)$. The attaching map is the concatenation of four loops, the images α, β, γ and δ of the sides AB, BC, CD and DA. Clearly $\gamma = \overline{\alpha}$ and $\delta = \overline{\beta}$. Therefore $H_1(f)([z]) = [\alpha + \beta + \overline{\alpha} + \overline{\beta}] = 0$. It now follows from the long exact sequence of the pair $(T, S^1 \vee S^1)$ that $H_1(T; R) \cong R^2$ and $H_2(T; R) \cong R$.

The other surfaces may also be obtained by adjoining a single 2-cell to a wedge of circles. In particular, $T \sharp T = \vee^4 S^1 \cup e^2$ may be obtained from a regular octagon in the hyperbolic plane by identifying sides in pairs. This construction generalizes further: $\sharp^g T \simeq \vee^{2g} S^1 \cup e^2$. (See the figure on the next page.)

Exercise 23. *Compute the homology of* $P^2(\mathbb{R}) = S^1 \cup_f D^2$*, where* $f : S^1 = \partial D^2 \to S^1$ *is the map sending* z *to* z^2*.*

[Hint: $H_1(f) : H_1(S^1; \mathbb{Z}) \to H_1(S^1; \mathbb{Z}) \cong \mathbb{Z}$ is multiplication by 2.]



Exercise 24. *Complex projective* n*-space may be constructed inductively, i.e.,* $P^{n+1}(\mathbb{C}) = P^n(\mathbb{C}) \cup_f D^{2n+2}$ *for some map* $f : S^{2n+1} \to P^n(\mathbb{C})$*. Prove by induction on* n *that* $H_q(P^n(\mathbb{C}); R) = 0$ *if* q *is odd or if* $q > 2n$*, while* $H_q(P^n(\mathbb{C}); R) \cong R$ *if* q *is even and* $0 \le q \le 2n$*.*

(You don't need to know much about f for this question!)

Exercise 25. *Compute the homology of the Klein bottle* $Kb = S^1 \vee S^1 \cup_f e^2$*, where* $f : S^1 = \partial D^2 \to S^1 \vee S^1$ *is the loop* $xyx^{-1}y$*.*

18. EXCISION – THE LEBESGUE COVERING LEMMA, SUBDIVISION

Let (X, A) be a pair of spaces and U a subset such that $\overline{U} \subseteq intA$.

Definition. Let $\mathcal{V}$ be an open cover of X. Then a singular q-simplex $\sigma : \Delta_q \to X$ is *small of order* $\mathcal{V}$ if $\sigma(\Delta_q) \subseteq U$ for some $U \in \mathcal{V}$.

We shall apply this definition to the cover $\{X \setminus \overline{U}, intA\}$, and show that any singular q-simplex in X is homologous to a sum of small q-simplices, obtained by subdivision. The subdivision will take place on the domain (standard q-simplex) and extend functorially by composition with maps. We shall use the fact that Δ_q is a compact metric space.

Lemma 3 (Lebesgue)**.** *If $\mathcal{V}$ is an open cover of the compact metric space M there is an $\epsilon > 0$ such that each subset of diameter at most ϵ is contained in some member of $\mathcal{V}$.*

Proof. For each $m \in M$ there is a $V \in \mathcal{V}$ such that $m \in V$, and hence there is an $\epsilon(m) > 0$ such that the open ball $B_{2\epsilon(m)}(m)$ is contained in V. Since $M = \cup_{m\in M} B_{\epsilon(m)}(m)$ and is compact there is a finite subset $F \subset M$ such that $M = \cup_{m\in F} B_{\epsilon(m)}(m)$. Then $\epsilon = \min\{\epsilon(m) \mid m \in F\}$ works. □

Definition. Let $A_0, \dots, A_q$ and B be points in $\mathbb{R}^n$, not necessarily distinct or affinely independent. The *linear* singular q-simplex $\alpha = (A_0, \dots, A_q)$ in $\mathbb{R}^n$ is defined by $\alpha(x_0, \dots, x_n) = \Sigma x_i A_i$, for all $(x_0, \dots, x_n) \in \Delta_q$. The *join* of B and $(A_0, \dots, A_q)$ is the the linear singular $(q+1)$-simplex

$$B * (A_0, \dots, A_q) = (B, A_0, \dots, A_q).$$

This is a singular version of a more geometric notion. If $n = p + q$, $B \subseteq \mathbb{R}^p \times \{O\} \subset \mathbb{R}^n$, $C \subseteq \{O\} \times \mathbb{R}^q \subset \mathbb{R}^n$ and $B \cap C = \emptyset$, the *join* of B and C is

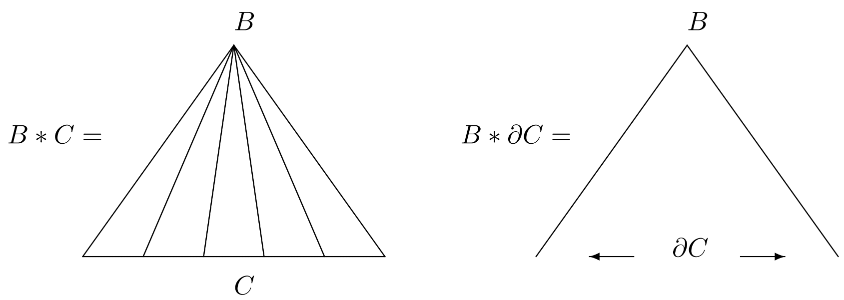
$$B * C = \{tb + (1-t)c \mid b \in B, c \in C, 0 \leq t \leq 1\}.$$

Our definition allows "degenerate" cases, where the vertices $\{B, A_0, \dots A_q\}$ lie in some q-dimensional hyperplane. (In particular, we allow $B \in C$.)

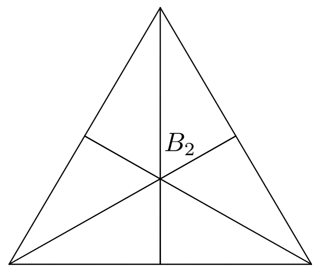
Joining extends to linear combinations of such simplices in an obvious way.

Lemma 4. *Let $C = \Sigma r_i \alpha_i$ be a linear singular q-chain in $\mathbb{R}^n$. If $q \geq 1$ then $\partial(B * C) = C - B * \partial C$. If $q = 0$ then $\partial(B * C) = C - (\Sigma r_i)B$.*

Proof. It suffices to assume that C is a q-simplex. The result is geometrically clear (see figure); the only point to check is that the signs are correct. □



Definition. The *barycentre* of Δ_q is $B_q = (\frac{1}{q+1}, \dots, \frac{1}{q+1})$.



Let $\delta_q = id_{\Delta_q}$, considered as a singular q-simplex in the space Δ_q.

We shall define the subdivision of δ_q inductively, extending the subdivision of its boundary by "joining" with B_q.

Let $Sd(\delta_0) = \delta_0$. Suppose $Sd(\delta_q)$ has been defined. We may extend Sd functorially to other singular q-simplices by $Sd(\sigma) = C_q(\sigma)(Sd(\delta_q))$, and then linearly to all singular q-chains. Hence $Sd(\partial\delta_{q+1}) = \Sigma_{i=0}^{i=q+1}(-1)^i Sd(\delta_q^{(i)})$, where $\delta_q^{(i)}$ is the i^{th} face of δ_{q+1}. Therefore we may define

$$Sd(\delta_{q+1}) = B_{q+1} * Sd(\partial\delta_{q+1}).$$

Lemma 5. *$\partial Sd = Sd\partial$.*

Proof. By functoriality it suffices to check the values on δ_q. We induct on q. The result is true for $q = 0$. Now $\partial Sd(\delta_q) = \partial(B * Sd(\partial\delta_q)) = Sd(\partial\delta_q) - B * \partial(Sd(\partial\delta_q))$. As $\partial(Sd(\partial\delta_q)) = Sd(\partial\partial\delta_q) = 0$, by the inductive hypothesis, this gives $\partial Sd(\delta_q) = Sd(\partial\delta_q)$. □

We must also check that subdivision preserves homology classes. To do this systematically we shall construct a "chain homotopy" operator $T : C_q(X) \to C_{q+1}(X)$ such that $Id - Sd = \partial T + T\partial$. Hence if z is a cycle

$$z - Sd(z) = \partial T(z) + T(\partial z) = \partial T(z)$$

is a boundary, and so $[z] = [Sd(z)]$. We again insist that T be functorial, and induct on q. Thus it suffices to define T for the standard simplex. Let

$$T(\delta_0) = 0 \quad \text{and} \quad T(\delta_q) = B_q * (\delta_q - Sd(\delta_q) - T(\partial\delta_q)), \quad \text{if} \quad q > 0.$$

(Here $B_q * \delta_q$ is the "degenerate" $(q+1)$-simplex whose image is just Δ_q). If σ is a singular q-simplex in X let $T(\sigma) = C_{q+1}(\sigma)T(\delta_q)$.

Lemma 6. *$Id - Sd = \partial T + T\partial$.*

Proof. As before, it suffices to check the values on δ_q, and induct on q. The result is true for $q = 0$. If $q > 0$ then $\partial T(\delta_q) + T(\partial(\delta_q)) =$

$= \partial(B_q * (\delta_q - Sd(\delta_q) - T(\partial\delta_q)) + T(\partial\delta_q)$

$= \delta_q - Sd(\delta_q) - T(\partial\delta_q) - B_q * (\partial\delta_q - \partial Sd(\delta_q) - \partial T(\partial\delta_q)) + T(\partial\delta_q)$

$= \delta_q - Sd(\delta_q) - B_q * (\partial\delta_q - Sd(\partial\delta_q) + T(\partial\partial\delta_q) + (Sd - Id)(\partial\delta_q))$

$= \delta_q - Sd(\delta_q)$. □

19. EXCISION – COMPLETION OF ARGUMENT

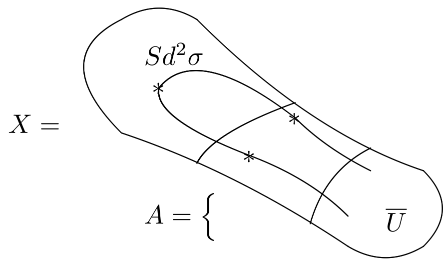
Finally we must check that on iterating the subdivision operator sufficiently many times we do indeed obtain sums of small simplices. We need the basic estimate: if σ is an affine q-simplex of diameter $d(\sigma)$ then each summand of $Sd(\sigma)$ has diameter at most $\frac{q}{q+1}d(\sigma)$.

Lemma 7. *Let $A_0, \dots A_q$ be the vertices of an affine q-simplex σ in $\mathbb{R}^N$. Then the diameter of σ is $diam(\sigma) = \max\{||A_i - A_j|| \mid 0 \leq i < j \leq q\}$.*

Proof. If $x \in \sigma$ then $x = \Sigma t_i A_i$, where $t_i \geq 0$ and $\Sigma t_i = 1$. Hence $||x - y|| = |\Sigma t_i(A_i - y)|| \leq \Sigma t_i ||A_i - y|| \leq \max\{||A_i - y|| \mid 0 \leq i \leq q\}$. If y is also in σ a similar argument gives $||x - y|| \leq \max\{||A_i - A_j|| \mid 0 \leq i < j \leq q\}$. □

Lemma 8. *Let σ be an affine q-simplex in $\mathbb{R}^N$ and σ' one of the simplices of $Sd(\sigma)$. Then $diam(\sigma') \leq \frac{q}{q+1} diam(\sigma)$.*

Proof. The vertices of σ' are the barycentres of faces of σ. The dimension of the corresponding face determines a natural ordering on these vertices, say $B_0 = A_0$, $B_1 = \frac{1}{2}(A_0 + A_1)$, $B_2 = \frac{1}{3}(A_0 + A_1 + A_2)$, $\dots$, $B_q = \frac{1}{q+1}(A_0 + \dots + A_q)$. If $r < s \leq q$ then $||B_r - B_s|| \leq \max\{||A_i - B_s|| \mid 0 \leq i \leq r\}$. Now if $i \leq r$ then $||A_i - B_r|| = ||\Sigma_{j=0}^{j=r}(A_i - A_j)||/(r+1) \leq \frac{r}{r+1} \max\{||A_i - A_j|| \mid 0 \leq j \leq r\} \leq \frac{q}{q+1} diam(\sigma)$. The lemma follows easily. □



Lemma 9. *Every homology class in $H_q(X, A)$ can be represented by a relative cycle which is a combination of singular simplices which are small of order $\{X \setminus \overline{U}, intA\}$.*

Proof. For each singular q-simplex $\sigma : \Delta_q \to X$ there is an $n_\sigma \geq 0$ such that $(Sd)^{n_\sigma}(\delta_q)$ is a linear combination of singular simplices which are small of order $\{\sigma^{-1}(X \setminus \overline{U}), \sigma^{-1}(intA)\}$, by the Lebesgue Covering Lemma and the estimate above. Given a relative q-cycle $z = \Sigma r_\sigma \sigma$, let $n = \max\{n_\sigma \mid r_\sigma \neq 0\}$. Then $(Sd)^n(z)$ is a linear combination of singular simplices which are small of order $\{X \setminus \overline{U}, intA\}$. Moreover $z - Sd(z) = \partial T(z) + T(\partial z)$ is a relative boundary. Hence $(Sd)^n(z)$ represents the same relative homology class as z. □

Theorem 10. *Let X be a space with a subspace A, and let $U \subseteq A$. If the closure of U is contained in the interior of A then the inclusion of $(X \setminus U, A \setminus U)$ into (X, A) induces isomorphisms on relative homology, $H_q(X \setminus U, A \setminus U; R) \cong H_q(X, A; R)$, for all q and any coefficients R.*

Proof. 1-1: Suppose z is a singular q-chain on $X \setminus U$ such that ∂z is a $(q-1)$-chain on $A \setminus U$ and that z is a relative boundary in (X, A), i.e., that there is a singular q-chain z' on A and a singular $q+1$-chain w on X such that $z = z' + \partial w$. Then $(Sd)^n(z) = (Sd)^n(z') + \partial(Sd)^n(w)$. If $(Sd)^n(w)$ is small of order $\{X \setminus \overline{U}, intA\}$, say $(Sd)^n(w) = w_{X \setminus \overline{U}} + w_A$ then $(Sd)^n(z) - \partial w_{X \setminus \overline{U}} = (Sd)^n(z') + \partial w_A$. Since the LHS is a chain on $X \setminus U$ and the RHS is a chain on A both sides are chains on $A - U$. Hence $(Sd)^n(z) = ((Sd)^n(z) - \partial w_{X \setminus \overline{U}}) + \partial w_{X \setminus \overline{U}}$ is a relative boundary in $(X \setminus U, A \setminus U)$.

onto: Similar. □

A *knot* is an embedding K of one sphere inside another, and a *link* is an embedding L of several disjoint spheres. These are usually studied through invariants of their complements. The purpose of the next exercise is to show that homology alone gives little information!

Let $K : S^n \to S^{n+k}$ be an embedding (homeomorphism onto its image). To avoid pathologies, we asume that the image of K has a product neighbourhood $N(K) \cong S^n \times D^k$. Let $\frac{1}{2}N(K)$ be a concentric, smaller neighbourhood. Let $X = S^{n+k} \setminus intN(K)$. Then X is a compact manifold with boundary $\partial X \cong \partial S^n \times D^k = S^n \times S^{k-1}$. Moreover the inclusion of X into $S^{n+k} \setminus \frac{1}{2}N(K)$ is a homotopy equivalence. (This is not immediately obvious, but should be plausible.)

Exercise. *Assume that $n, k > 0$. Use excision (and homotopy invariance) to show that if $i < n + k - 1$ then $H_i(X) = 0$ unless $i = 0$ or $k - 1$, in which case $H_i(X; R) \cong R$.*

[Hint. $S^{n+k} = \partial D^{n+k+1} = \partial(D^{n+1} \times D^k)$, so $S^{n+k} = S^n \times D^k \cup D^{n+1} \times S^{k-1}$.]

In fact, $H_i(X; R)$ is also 0 for all $i \geq n + k - 1$. This follows easily from the long exact sequence of the pair (S^{n+k}, X) if $i \neq n + k - 1$, since $X \subset S^{n+k}\setminus\{P\} \subset S^{n+k}$ (where P is any point in $S^{n+k} \setminus X$). To see that $H_{n+k-1}(X; R)$ is also 0 requires an additional fact: if M is an m-manifold with connected boundary, the inclusion of ∂M into M induces the trivial homomorphism on H_{m-1}.

What happens if $n = 0$ or $k = 0$? Can you see what happens for an embedding $L : \mu S^n \to S^{n+k}$? (The result holds for embeddings of μS^n (without such product neighbourhoods) but this extension is usually done via Alexander duality.)

20. Spectral theory for orthogonal matrices

Let $\langle x, y\rangle = \Sigma x_i y_i$ be the standard inner product on $\mathbb{R}^n$ and $\langle v, w\rangle = \Sigma v_i \bar{w}_i$ be the standard inner product on $\mathbb{C}^n$. If U is a subspace of $\mathbb{R}^n$ its *orthogonal complement* is $U^\perp = \{x \in \mathbb{R}^n \mid \langle x, u\rangle = 0 \ \forall u \in U\}$. The inner product on $\mathbb{R}^n$ induces inner products on subspaces, and $\mathbb{R}^n \cong U \perp U^\perp$ is the orthogonal direct sum. (Similarly for subspaces of $\mathbb{C}^n$.)

A matrix $A \in GL(n, \mathbb{R})$ is *orthogonal* if $\langle Ax, Ay\rangle = \langle x, y\rangle$ for all $x, y \in \mathbb{R}^n$. A matrix $B \in GL(n, \mathbb{C})$ is *unitary* if $\langle Bv, Bw\rangle = \langle v, w\rangle$ for all $v, w \in \mathbb{C}^n$. Any real matrix $A \in GL(n, \mathbb{R})$ acts on $\mathbb{C}^n$ in a natural way, and so we may view $GL(n, \mathbb{R})$ as a subgroup of $GL(n, \mathbb{C})$. Under this identification, the orthogonal matrices are exactly the real unitary matrices.

The advantage of working over the complex numbers is that we can always find eigenvalues and eigenvectors. Let u be an eigenvector of the unitary matrix B corresponding to an eigenvalue λ and let $U = \mathbb{C}u$. We may assume that $\langle u, u\rangle = 1$. Since B is unitary it also maps $U^\perp$ to itself, and so preserves the direct sum decomposition $\mathbb{C}^n \cong U \perp U^\perp$. Hence we may prove by induction on n that $\mathbb{C}^n$ has an orthonormal basis consisting of eigenvectors for B. Let V be the matrix whose columns are these eigenvectors. Then V is unitary (since the columns are orthonormal) and $BV = V\Lambda$, where Λ is a diagonal matrix whose diagonal entries are the eigenvalues of B. (It is easy to see that eigenvectors corresponding to distinct eigenvalues must be orthogonal.)

We now carry this argument over to the reals. Let A be an orthogonal matrix, and let v be an eigenvector in $\mathbb{C}^n$ for A, corresponding to the eigenvalue $\lambda \in \mathbb{C}$. We may write $v = v_1 + iv_2$ and $\lambda = c + is$, where $v_1, v_2 \in \mathbb{R}^n$ and $c, s \in \mathbb{R}$. On comparing real and imaginary parts in the equation $Av = \lambda v$ we find that $Av_1 = cv_1 - sv_2$ and $Av_2 = sv_1 + cv_2$. If $s = 0$ then $\lambda = \pm 1$ and each of the vectors v_1 and v_2 is a real eigenvector. Otherwise v_1 and v_2 are perpendicular. [Expand out the equations $(Av_1, Av_1) = (v_1, v_1)$ and $(Av_1, Av_2) = (v_1, v_2)$, to get $(c^2 - s^2)(v_1, v_2) = (v_1, v_2)$]. The matrix A acts as a rotation on the 2-dimensional subspace of $\mathbb{R}^n$ spanned by $\{v_1, v_2\}$. Since A is orthogonal it also maps the orthogonal complement of this subspace to itself.

Thus arguing by induction on n, we can find a basis for $\mathbb{R}^n$ with respect to which A is block-diagonal, the diagonal blocks being either 1×1 blocks $[\pm 1]$ or 2×2 rotation matrices. Equivalently: there is an orthogonal matrix P such that PAP^{-1} is block diagonal.

(Note also that $diag[-1, -1]$ is a rotation matrix, and $det(A) = (-1)^s$, where s is the number of -1s on the diagonal.)

The orthogonal group $O(n + 1)$ acts transitively on on the lines through O in $\mathbb{R}^n$, while the unitary group $U(n + 1)$ acts transitively on the lines through O in $\mathbb{C}^n$. Thus $P^n(\mathbb{R}) = O(n + 1)/O(n)$ and $P^n(\mathbb{C}) = U(n + 1)/U(n)$.

21. SELF-MAPS OF S^n (1): ORTHOGONAL HOMEOMORPHISMS

An $(n + 1) \times (n + 1)$ orthogonal matrix A determines a self-homeomorphism of $S^n = \{(x_0, \dots, x_n) \in \mathbb{R}^{n+1} \mid \ ||x|| = 1\}$, since $||Ax|| = ||x||$.

Definition. The *degree* of a self-map $f : S^n \to S^n$ is the integer $deg(f)$ such that $H_n(f)([z]) = deg(f)[z]$ for all $[z] \in H_n(S^n; \mathbb{Z}) \cong \mathbb{Z}$.

It is easy to see that the degree is multiplicative: $deg(fg) = deg(f)deg(g)$. In particular, if f is a homotopy equivalence then $deg(f) = \pm 1$.

Theorem 11. *If $A \in O(n + 1)$ then $deg(A) = det(A)$.*

Proof. There is an orthogonal matrix P such that PAP^{-1} is block-diagonal. Since $deg(A) = deg(PAP^{-1})$ we may assume that A is block-diagonal. Suppose first that $det(A) = +1$. Then there must be an even number of -1s on the diagonal, and so these may be grouped in pairs, corresponding to rotations through π. Thus $A = diag[R(\theta_1), \dots, R(\theta_k), 1, \dots, 1]$. We may define a homotopy A_t from $A_0 = I$ to $A_1 = A$ through orthogonal matrices by $A_t = diag[R(t\theta_1), \dots, R(t\theta_k), 1, \dots, 1]$ for $0 \le t \le 1$. Hence $deg(A) = deg(I) = 1$.

Now let $M_{n+1} = diag[-1, 1, \dots, 1]$ be the orthogonal matrix corresponding to reflection in the first coordinate. We shall show that $deg(M_{n+1}) = -1 = det(M_{n+1})$. The reflection M_{n+1} preserves the hemispheres $D^n_\pm$ and restricts to M_n on the equator S^{n-1}. If $n > 1$ there are isomorphisms $H_n(S^n) \cong H_n(S^n, D^n_-) \cong H_n(D^n_+, S^{n-1}) \cong H_{n-1}(S^{n-1})$, and M_{n+1} induces commuting diagrams involving these isomorphisms. (We are using the naturality of the connecting homomorphisms here!) Hence $deg(M_{n+1}) = deg(M_n)$ for all $n \ge 2$, by an easy induction. If $n = 1$ there is an exact sequence

$$0 \to H_1(S^1) \to H_0(S^0) \to H_0(D^1_+) \to 0.$$

The image of $H_1(S^1)$ in $H_0(S^0)$ is generated by the 0-cycle $[+1] - [-1]$. Since M_1 interchanges $+1$ and -1 in S^0 it follows that $deg(M_2) = -1$.

Finally, if A is orthogonal and $det(A) = -1$ then $det(AM_{n+1}) = +1$ so $deg(A) = deg(AM^2_{n+1}) = deg(AM_{n+1})deg(M_{n+1}) = -1 = det(A)$. $\square$

Example. Let $a_n = -I_{n+1} : S^n \to S^n$ be the antipodal map, given by $a_n(x) = -x$ for all $x \in S^n$. Then $deg(a_n) = det(-I_{n+1}) = (-1)^{n+1}$. Hence a_n is homotopic to I if and only if n is odd.

22. FIXED POINTS

Let $f : X \to X$ be a map. A point x is a fixed point if $f(x) = x$. For instance, I fixes every point, $a_n : S^n \to S^n$ fixes nothing. If $f : S^n \to S^n$ has no fixed points then it is homotopic to a_n. (Normalize a linear homotopy.)

In particular, every self map of S^{2n} of degree $+1$ has a fixed point.

Exercise 26. *Let $f : D^n \to D^n$ be a continuous function. Show that f has a fixed point, i.e. that $f(z) = z$ for some z in D.*

[Hint: Suppose not. Then the line from $f(z)$ through z is well defined. Let $g(z)$ be the point of intersection of this line with S^{n-1} such that z lies between $f(z)$ and $g(z)$. Show that g is a continuous function from D^n to S^{n-1} such that $gi = id_{S^{n-1}}$, where $i : S^{n-1} \to D^n$ is the natural inclusion (i.e., $g(z) = z$ for all z in S^{n-1}). Show that no such map can exist.]

Exercise 27. *Let $f : S^n \to S^n$ be a map which is homotopic to a constant map. Show that f fixes some point of S^n (i.e., $f(x) = x$) and also sends some other point to its antipodal point (i.e., $f(y) = -y$).*

Exercise 28. *Let $f : S^{2n} \to S^{2n}$ have no fixed point. Show that there is a point $x \in S^{2n}$ such that $f(x) = -x$.*

Exercise 29. *Let G be a group of homeomorphisms of S^{2n} such that each non-identity element of G has no fixed point. Show that $|G| \leq 2$.*

The connection between homology and fixed points extends to more general spaces via the Lefshetz Fixed Point Formula. If f has no fixed point then

$$Lef(f) = \Sigma(-1)^q tr(H_q(f; F)) = 0,$$

for any field coefficients F. Note that $Lef(id_X) = \chi(X)$.

23. VECTOR FIELDS

A vector field on a smooth manifold M is a continuous family of tangent vectors. Thus if $M = S^n$ is the unit n-sphere in $\mathbb{R}^{n+1}$ a vector field is a map $v : S^n \to \mathbb{R}^{n+1}$ such that $x \bullet v(x) = 0$ for all $x \in S^n$. If v is a nowhere 0 vector field we may normalize it to get $\widehat{v}$, given by $\widehat{v}(x) = v(x)/||v(x)||$ for all $x \in S^n$. The function $F_t(x) = cos(\pi t)x + sin(\pi t)\widehat{v}(x)$ then gives a homotopy from $F_0 = I$ to $F_1 = a_n$. In particular, n must be odd. Conversely, if n is odd then $v(x_0, \ldots, x_n) = (x_1, -x_0, \ldots, x_n, -x_{n-1})$ is a (nowhere 0) unit vector field on S^n.

Much deeper arguments show that S^n admits n everywhere linearly independent vector fields $\Leftrightarrow n = 0, 1, 3$ or 7. This has an algebraic consequence. If R^{n+1} has a bilinear multiplication such that products of nonzero elements are nonzero, then normalizing the product of elements of S^n gives a continuous multiplication on S^n. We may use this to translate a basis for the tangent space at one point to a continuous family of bases. Hence the only finite dimensional real division algebras are $\mathbb{R}$, $\mathbb{C}$, $\mathbb{H}$ and $\mathbb{O}$ (the Cayley octonions), of dimensions 1, 2, 4 and 8.

FACT. A closed n-manifold M supports a nowhere 0 vectorfield if and only if $\chi(M) = 0$. (This can be related to the Lefshetz Fixed Point Formula. If M has a nowhere 0 vectorfield then we may use the vector field to deform id_M slightly to a fixed-point free self map, and so $\chi(M) = Lef(id_M) = 0$.) It follows from Poincaré duality that $\chi(M) = 0$ if N is odd.

The torus and Klein bottle clearly have such vector fields. To see that no other surfaces have such vector fields, we consider the larger class of vector fields which vanish at only finitely many points. To each such point we assign a local "Hopf index" ± 1. Given two such vector fields $\mathbf{v}$ and $\mathbf{w}$ on a surface S, we use the linear homotopy of vector fields $(1-t)\mathbf{v} + t\mathbf{w}$ to show that the sum of these local indices depends only on the surface, and not on the vector field. An explicit construction gives a vector field which is 0 only at the vertices of the barycentric subdivision of a triangulation, and for which the integer is $\chi(S)$. Thus if $\chi(S) \neq 0$ every vector field must have 0s. See *Selected Applications of Geometry to Low-Dimensional Topology*, by Freedman and Luo [Science Library 513.83/325] for more details.

24. self maps of S^n (2): suspension

We shall define an addition on $[S^n; S^n]$ which makes this set into an abelian group. With the product corresponding to composition of functions it becomes a ring, and we shall show that $deg : [S^n; S^n] \to \mathbb{Z}$ is an isomorphism of rings. It is easy to see that the degree is multiplicative: $deg(fg) = deg(f)deg(g)$. In particular, if f is a homotopy equivalence then $deg(f) = \pm 1$. (The hardest part is showing that deg is 1-to-1, and we may only sketch this.)

The *suspension* of a space X is the space $SX = [-1, 1] \times X/\sim$, where $(-1, x) \sim (-1, x')$ and $(1, x) \sim (1, x')$ for all $x, x' \in X$. (Thus each end of the cylinder has been crushed to a point $\pm\infty$.)

In order that the constructions used below be well-defined we require that our spaces have "basepoints", and that all functions, homotopies, etc., respect these basepoints. If $x_0 \in X$ and $y_0 \in Y$ are basepoints let $[X; Y]_*$ denote the set of homotopy classes of basepoint preserving maps $f : X \to Y$ (i.e., such that $f(x_0) = y_0$ and the homotopies are constant on the basepoints). Give SX the basepoint $[0, x_0]$, and let I_0 be the subspace $[-1, 1] \times \{x_0\}$. This is an interval through the basepoint and connecting $\pm\infty$. In the pointed category $((Hot_*))$ it is more natural to replace SX by the *reduced* suspension $\Sigma X = SX/I_0$, in which I_0 has been crushed to a point, and which thus has a natural basepoint (the image of I_0). The natural map from SX to ΣX is a homotopy equivalence, and composition with this map gives a bijection $[\Sigma X; Y]_* \to [SX; Y]_*$. (It can be shown that ΣS^n is homeomorphic to $SS^n = S^{n+1}$. This is the most important case for us.)

Let $[t, x]$ denote the image of (t, x) in ΣX for all $-1 \leq t \leq 1$ and $x \in X$.

If $f : X \to Y$ is a map then $\Sigma f : \Sigma X \to \Sigma Y$ is the map defined by $\Sigma f([t, x]) = [t, f(x)]$ for all $-1 \leq t \leq 1$ and $x \in X$.

If $f : \Sigma X \to Z$ is a map then $\bar{f} : \Sigma X \to Z$ is the map defined by $\bar{f}([t, x]) = f([-t, x]$ for all $-1 \leq t \leq 1$ and $x \in X$.

Let ΣX^+ and ΣX^- be two copies of ΣX, and let $\Sigma X^+ \vee \Sigma X^-$ be the one-point union of these spaces, with basepoints identified. Then there is a "pinch" map $p : \Sigma X \to \Sigma X^+ \vee \Sigma X^-$, given by $p([t, x]) = [2t - 1, x]^+$ if $t \geq 0$ and $p([t, x]) = [2t + 1, x]^-$ if $t \leq 0$.

If $f, g : \Sigma X \to Z$ are maps then they define a map $f \vee g : \Sigma X^+ \vee \Sigma X^- \to Z$. We set $f + g = (f \vee g)p : \Sigma X \to Z$. It is straightforward to verify that if $f \sim f'$ and $g \sim g'$ then $f + g \sim f' + g'$, so we get an "addition" on $[\Sigma X; Z]_*$. (The notation is

misleading, as in general $g+f$ is not homotopic to $f+g$.) Let $c:\Sigma X\to Z$ denote the constant map (with value the basepoint of Z).
Claim:
(i) $f+c\sim f\sim c+f$
(ii) $f+\bar{f}\sim c\sim \bar{f}+f$
(iii) $((f+g)+h)\sim (f+(g+h)$

We shall prove (i) and (ii) by giving explicit formulae for the case $f=id_{\Sigma X}$. (The general case follows by composition with $f:\Sigma X\to Z$.)

Let $h_s([t,x])=[1+2s(|t|-1),x]$ and $k_s([t,x]=[t+s(|t|-1),x]$, for all $0\leq s\leq 1$, $-1\leq t\leq 1$ and $x\in X$. Then h_s is a homotopy from $h_0=c$ to $h_1=id_{\Sigma X}+\overline{id_{\Sigma X}}$, while k_s is a homotopy from $k_0=id_{\Sigma X}$ to $k_1=id_{\Sigma X}+c$.

Check that a similar argument establishes the associativity of the operation.

Thus $[\Sigma X;Y]_*$ is a group, with identity represented by the constant map c and with $\bar{f}$ representing the inverse of f.
Claim: $[\Sigma^2X;Y]$ is an abelian group, and the natural "forget basepoints" function from $[\Sigma^2X;Y]_*$ to $[S^2X;Y]$ is a bijection.

25. DEGREE

In this section we shall outline why *deg* defines an isomorphism from $[S^n;S^n]$ to $\mathbb{Z}$. It is easy to see that *deg* is onto, and not hard to show that it is a homomorphism, but our argument for injectivity rests on ideas from beyond this course, and we shall not give the full details.

Exercise 30. *Show that if $f:S^n\to S^n$ then $deg(Sf)=deg(f)$.*

[Hint: adapt the inductive step in the proof of Theorem 11. We have used the *un*reduced suspension here, but it is also true that $deg(\Sigma f)=deg(f)$.]

In particular, if $f_k:S^1\to S^1$ is the k^{th} power map then $\Sigma^{n-1}f_k:S^n\to S^n$ has degree k, and so *deg* is onto, for all $n\geq 1$.

In order to see that *deg* is a homomorphism we must calculate $H_n(f\vee g)$ and $H_n(p)$, where $p:S^n\to W=S^n_+\vee S^n_-$ is the pinch map, which crushes the equator to a point. Let $j_\pm:S^n\to S^n_\pm$ be the inclusions of the factors of W, and let $q_\pm:W\to S^n$ be projections such that $q_+j_+=q_-j_-=id_{S^n}$ and $q_+j_-=q_-j_+$ is constant. Then $q_\pm\sim id_{S^n}$, $(f\vee g)j_+\sim f$ and $(f\vee g)j_-\sim g$.

Fix a generator $[S^n]$ for $H_n(S^n)$. It is easy to see that $H_n(W)\cong\mathbb{Z}\oplus\mathbb{Z}$. We may use the maps $j_\pm$ and $q_\pm$ to identify an explicit basis The homomorphism $\lambda:\mathbb{Z}\oplus\mathbb{Z}\to H_n(W)$ given by $\lambda(a,b)=aH_n(j_+)([S^n])+bH_n(j_-)([S^n])$ is onto, and $(H_n(q_+),H_n(q_-))\lambda=id_{\mathbb{Z}\oplus\mathbb{Z}}$, so λ is an isomorphism. We then see that $H_n(p)(c[S^n])=H_n(j_+)(c[S^n])+H_n(j_-)(c[S^n])=\lambda(c,c)$ is the diagonal map, while

$$H_n(f\vee g)(\lambda(a,b))=(adeg(f)+bdeg(g))[S^n].$$

Hence $H_n(f+g)([S^n])=H_n(f\vee g)\lambda(1,1)=(deg(f)+deg(g))[S^n]$, and so

$$deg(f+g)=deg(f)+deg(g).$$

We shall only sketch a geometric argument for injectivity.
Claim: any map $f:S^n\to S^n$ can be approximated by a smooth (C^∞) map, and sufficiently close approximations are homotopic to f. If f is C^∞ then there is a dense open subset $U\subseteq S^n$ such that for all $P\in U$ the preimage $f^{-1}(P)$ is finite, and for all $Q\in f^{-1}(P)$ the differential $Df(Q)$ is invertible.

Let $\epsilon_Q=sign(\det(Df(Q))=\pm 1$.

Claim: $deg(f) = \Sigma_{f(Q)=P}\epsilon_Q$.
Claim: if $deg(f) = 0$ then we may homotope f so that there is a $P \in S^n$ such that $f^{-1}(P) = \emptyset$.

Since $S^n \setminus \{P\} \cong \mathbb{R}^n$ it follows that f is homotopic to a constant map. Thus deg is injective.

We may use self-maps of spheres to construct finite complexes with arbitrary (finitely generated) homology. Assume $n \geq 2$ and let $X_k = S^{n-1} \cup_{f_k} e^n$, where $f_k : S^{n-1} \to S^{n-1}$ has degree k. Then X_k is path-connected and $H_q(X_k) = 0$ unless $q = 0$, $n-1$, or n. Moreover $H_{n-1}(X;R) \cong R/kR$ and $H_n(X;R) = \mathrm{Ker}(: r \mapsto kr)$.

In particular, given integers $m, n > 0$, there is a connected finite cell complex X such that $H_m(X;\mathbb{Z}) \cong Z/nZ$ and $H_q(X;\mathbb{Z}) = 0$ if $q \neq 0$ or m.

Exercise 31. *Let H_i be a finitely generated abelian group, for $1 \leq i \leq n$. Show that there is a connected finite cell complex X (with all cells of dimension at most $n+1$) such that $H_q(X;\mathbb{Z}) \cong H_q$, for $1 \leq q \leq n$.*

26. The Hurewicz homomorphism and homology in low degrees

Let X be a space, with basepoint $*$. We may choose generators $[S^n]$ for $H_n(S^n;\mathbb{Z})$, for each $n \geq 1$, compatibly with suspension. If $f : S^n \to X$ is a based map then $f_*[S^n] = H_n(f)([S^n]) \in H_n(X;\mathbb{Z})$ depends only on the homotopy class of f. Let $g : S^n \to X$ be another based map. Then

$$(f+g)_*[S^n] = (f \vee g)_* p_*[S^n] = (f \vee g)_*(j_{+*}[S^n] + j_{-*}[S^n]) = f_*[S^n] + g_*[S^n].$$

Thus we obtain a homomorphism $hwz_n : \pi_n(X) = [S^n, X]_* \to H_n(X;\mathbb{Z})$, called the *Hurewicz homomorphism in degree n*.

Degree 1. When $n = 1$ the fundamental group $\pi = \pi_1(X)$ need not be abelian, and so hwz_1 maps the commutator subgroup $\pi' = [\pi, \pi]$ to 0, and thus induces a homomorphism from the abelianization $\pi^{ab} = \pi/\pi'$ to $H_1(X;\mathbb{Z})$. The Hurewicz Theorem asserts that if X is path-connected then hwz_1 is onto, with kernel π', so $\pi^{ab} \cong H_1(X;\mathbb{Z})$. (See Part II of these notes for a proof.)

Degree 2 or more. Each homology class in $H_2(X;\mathbb{Z})$ may be represented by a "singular surface" in X (i.e., a map $f : F \to X$, where F is a closed orientable surface). To see this, note that a singular 2-cycle is a formal sum of maps of triangles into X whose algebraic boundary is 0. The boundary condition implies that the edges of the triangles may be matched in pairs, on which the restrictions of the defining maps agree. We may glue the triangles together to get a closed surface, and the maps together give a map of this surface into X. It can be shown that two such singular surfaces f and f' represent the same homology class if and only if they together extend across some compact orientable 3-manifold W with boundary the disjoint union of F and $-F'$, i.e., there is a map $F : W \to X$ which restricts to f and f' on the boundaries. (The sign refers to an orientation condition.) There is a related "geometric" interpretation of the higher homology groups, involving maps of polyhedral manifolds (with "controlled" singularities of lower dimension). In the analytic context, a Riemannian metric on a smooth manifold determines canonical "harmonic" forms representing real cohomology classes (Hodge theory).

In every dimension $q \geq 1$ there is a function from $\pi_q(X) = [S^q; X]$ to $H_q(X;\mathbb{Z})$, sending $[f] \in \pi_q(X)$ to $H_q(f;\mathbb{Z})([S^q])$, where $[S^q]$ is a fixed generator for $H_q(S^q;\mathbb{Z})$. The Hurewicz Theorem in degree $q > 1$ asserts that this is a homomorphism, and if X is path-connected and all maps from S^k to X are homotopic to constant maps, for all $k < q$, then $H_k(X;\mathbb{Z}) = 0$ for $1 \leq k < q$ and hwz_q is an isomorphism. Thus the Hurewicz homomorphism relates the "additive", homological invariants to the "multiplicative" homotopy-group invariants.

27. COHOMOLOGY

If we consider the linear duals of our chain complexes, we obtain a more powerful invariant, the *cohomology ring* of a space. As we do not have time to do justice to this, I shall just sketch the easy bits here, and indicate in the final section below an alternative construction of the real cohomology ring for open subsets of $\mathbb{R}^n$, using differential forms.

Given a chain complex C_*, we define the associated cochain complex C^* by $C^q = Hom_R(C_q, R)$, with codifferential $\delta^q(f) = f \circ \partial_{q+1}$. The cohomology modules $H^q(C^*)$ are the quotients $\mathrm{Ker}(\delta^q)/\mathrm{Im}(\delta^{q-1})$. On applying this construction to the singular chain complexes of (pairs of) spaces, we obtain *contravariant* functors: if $f : (X, A) \to (Y, B)$ then $H^q(f) : H^q(Y, B) \to H^q(X, A)$. A pair of spaces determines a long exact sequence of cohomology, and cohomology satisfies homotopy and excision. The cohomology of a point is $H^*(\{P\}; R) = R, 0, 0, \ldots$.

In degree 0 the cohomology module $H^0(X; R)$ may be identified with the R-valued functions on X which are constant on components. (This is the first indication that cohomology may be related to functions and differential forms, since a function f on an open subset of $\mathbb{R}^n$ is locally constant if and only if $df = 0$.)

Let $ev : H^q(C^*; R) \to Hom_R(H_q(C_*; R), R)$ be the *evaluation* homomorphism, defined by $ev([f])([c]) = f(c)$. The quotient $C_q/Z_q \cong B_{q-1}$ is a submodule of the free module C_{q-1}. Therefore C_q/Z_q is a free module if $R = \mathbb{Z}$ or is a PID. (This is a standard algebraic fact, but is not trivial.) Hence any homomorphism from Z_q to R extends to a homomorphism from C_q to R. It follows that ev is an epimorphism. It is an isomorphism if R is a field.

The ring structure arises ultimately from the diagonal map $\Delta_X : X \to X \times X$. The ring $H^*(X; R)$ is a graded R-algebra, and is graded-commutative: $\alpha \cup \beta = (-1)^{pq}\beta \cup \alpha$. (Here $\cup$ denotes the product.) In particular, $H^*(S^n; \mathbb{Z}) \cong \mathbb{Z}[\eta_n]/(\eta_n^2)$, where η_n generates $H^n(S^n; \mathbb{Z})$.

Example. Let $h : S^3 \to S^2 = P^1(\mathbb{C})$ be the "Hopf fibration", given by $h(u, v) = [u : v]$ for all $(u, v) \in S^3$, and let $c : S^3 \to S^2$ be the constant map. Then $S^2 \vee S^4 = S^2 \cup_c e^4$ and $P^2(\mathbb{C}) = S^2 \cup_h e^4$. It is easy to see that $S^2 \vee S^4$ and $P^2(\mathbb{C})$ have isomorphic homology and cohohomology groups. However their cohomology *rings* are not isomorphic. In fact $H^*(S^2 \vee S^4; \mathbb{Z}) \cong \mathbb{Z}[\eta_2, \eta_4]/(\eta_2, \eta_4)^2$ and $H^*(P^2(\mathbb{C}); \mathbb{Z}) \cong \mathbb{Z}[\xi_2]/(\xi_2^3)$. Hence these spaces are not homotopy equivalent. In particular, h is not homotopic to a constant map, although $H_*(h) = H_*(c)$.

It is not hard to show directly that $ev : H^1(X; \mathbb{Z}) \to Hom(H_1(X; \mathbb{Z}), \mathbb{Z})$ is an isomorphism. The map from $[X; S^1]$ to $H^1(X; \mathbb{Z})$ given by $f \mapsto f^*\eta_1$ is also an isomorphism. In the second part of this course we shall see that if X is path-connected (and based) then $H_1(X; \mathbb{Z})$ is the abelianization of the fundamental group $\pi = \pi_1(X, *) = [S^1; X]_*$. Thus $H^1(X; \mathbb{Z}) \cong Hom(\pi, \mathbb{Z})$, and so $[X; S^1]$ may be viewed as being dual to $[S^1; X]_*$.

Cohomology groups in higher degrees (and with other coefficients) may also be identified with homotopy classes of maps into "classifying spaces" such as S^1, but this requires much more work.

28. TENSOR PRODUCTS

Let R be a commutative ring, M and N two R-modules. Let $R^{(M\times N)}$ be the free R-module with basis the *set* $M\times N$. Then $M\otimes_R N$ is the quotient of $R^{(M\times N)}$ by the submodule generated by the elements $r[m,n]-[rm,n]$, $r[m,n]-[m,rn]$, $[m+m',n]-[m,n]-[m',n]$ and $[m,n+n']-[m,n]-[m,n']$, for all $r\in R$, $m,m'\in M$ and $n,n'\in N$. Let $m\otimes n$ denote the image of $[m,n]$ in $M\otimes_R N$. The obvious function from $M\times N$ to $M\otimes_R N$ which sends (m,n) to $m\otimes n$ is bilinear. This is in fact the universal bilinear function; if P is another R-module and $b:M\times N\to P$ is bilinear then there is an unique linear map $\tilde{b}:M\otimes_R N\to P$ such that the obvious diagram commutes.

This construction and those that follow are suitably functorial.

Examples: R a field. $R^m\otimes_R R^n\cong R^{mn}$.

$R=\mathbb{Z}$. $(Z/mZ)\otimes_{\mathbb{Z}}(Z/nZ)\cong Z/(m,n)Z$.

Let $M^*=Hom_R(M,R)$. Then $M\otimes_R N^*\cong Hom_R(N,M)$, via the isomorphism $m\otimes\nu\mapsto(:n\mapsto\nu(n)m)$ (at least if M and N are finitely generated free modules).

We may iterate the process of forming tensor products. (The construction is associative.) In particular we may define tensor powers $\otimes^k M$ inductively, with $\otimes^0 M=R$, $\otimes^1 M=M$ and $\otimes^{k+1}M=(\otimes^k M)\otimes_R M$. The direct sum $\otimes^* M=\oplus_{k\geq 0}\otimes^k M$ is a noncommutative R-algebra under the obvious multiplication.

Example: R a field. The Künneth Theorem computes the homology of a product $H_t(X\times Y;F)\cong\oplus_{p+q=t}(H_p(X;F)\otimes H_q(Y;F))$. (There is a similar result for cohomology.)

In general $m\otimes m'\neq m'\otimes m$. However the set of all elements of $M\otimes_R M$ of the form $m\otimes m'-m'\otimes m$ generates a 2-sided ideal in $\otimes^*M$ and the quotient is a commutative R-algebra $\odot^*M$, the *symmetric algebra* on M. If $M\cong R^d$ then $\odot^*M$ is isomorphic to the polynomial ring in d variables over R.

We are more interested in the *exterior algebra* $\wedge^*M$. This is the quotient of $\otimes^*M$ by the ideal generated by all elements of $\otimes^2 M$ of the form $m\otimes m$. This quotient is graded-commutative, i.e., $\omega\wedge\xi=(-1)^{pq}\xi\wedge\omega$ if $\omega\in\wedge^pM$ and $\xi\in\wedge^qM$.

If $M\cong R^d$ then $\wedge^kM$ has rank $\binom{d}{k}$. In particular, $\wedge^0M=R$, $\wedge^1M=M$, $\wedge^dM\cong R$ and $\wedge^kM=0$ if $k>d$. Hence $\wedge^*M=\oplus_{k\geq0}\wedge^k M$ has rank 2^d. An ordered basis $\{m_1,\dots,m_d\}$ for M determines an (ordered) basis for $\wedge^kM$. In particular, it determines a generator $m_1\wedge\dots\wedge m_d$ for $\wedge^dM\cong R$. If $A:M\to M$ is R-linear then it induces endomorphisms $\wedge^kA$ of the exterior powers, and $\wedge^d(A)=det(A)$.

29. DE RHAM FOR OPEN SUBSETS OF $\mathbb{R}^n$

Let X be an open subset of $\mathbb{R}^n$. Let $S=C^\infty(X,\mathbb{R})$ be the ring of smooth functions on X. Let $\Omega^1(X)$ be the free S-module with basis $\{dx_1,\dots,dx_n\}$. This is the module of differential 1-forms on X. Let $\Omega^0(X)=S$ and $\Omega^k(X)=\wedge_k\Omega^1(X)$ for $k\geq 2$. Then $\Omega^n(X)\cong S.dx_1\wedge\dots\wedge dx_n$ and $\Omega^k(X)=0$ if $k>n$.

We shall define differentials $d:\Omega^k(X)\to\Omega^{k+1}(X)$ by means of partial derivatives. For $f\in S$ let $df=\Sigma f_i dx_i$ where $f_i=\partial f/\partial x_i$. Let $d(dx_i)=0$ for all i and extend by the Leibnitz rule: $d(\omega\wedge\xi)=(d\omega)\wedge\xi+(-1)^p\omega\wedge d\xi$, if $\omega\in\Omega^p(X)$, and then by linearity (over the reals $\mathbb{R}$). Then d is well defined and $dd=0$, essentially by equality of mixed partial derivatives. The De Rham cohomology of X is $H^q_{DR}(X)=\mathrm{Ker}(d:\Omega^q\to\Omega^{q+1})/d\Omega^{q-1}$. ("Closed" q-forms modulo "exact"

q-forms.) In particular, $H^0_{DR}(X) = \mathrm{Ker}(d : S \to \Omega^1(X))$ is the ring of locally constant functions on X.

Wedge-product of forms gives rise to a graded-commutative multiplication of cohomology classes.

Given a smooth path $\gamma : [0, 1] \to X$ and a 1-form $\omega = \Sigma g_i dx_i$ on X we may define $\int_\gamma \omega = \int_0^1 \Sigma g_i(\gamma(t))\dot{\gamma}_i(t)dt$. Then (Stokes' Theorem):

1) $\int_\gamma df = f(\gamma(1)) - f(\gamma(0))$.

2) if then $\int_\gamma \omega = \int_F d\omega$

Now $H_1(X; \mathbb{R}) \cong \{\gamma \mid \gamma(0) = \gamma(1)\}/\sim$, where $\gamma_1 \sim \gamma_2$ if $\gamma_1 \cup \bar{\gamma}_2$ together bound an oriented surface in X. (Here the overbar denotes a reversal of orientation.) Thus integration gives a pairing $H_1(X; \mathbb{R}) \times H^1_{DR}(X) \to \mathbb{R}$, sending $([\gamma], [\omega])$ to $\int_\gamma \omega$. This pairing is nondegenerate and so $H^1_{DR}(X) \cong H_1(X; \mathbb{R})^* \cong H^1(X; \mathbb{R})$. More generally, $H^q_{DR}(X) \cong H^q(X; \mathbb{R})$ (for reasonable open subsets X).

Any finite cell complex is homotopy equivalent to an open subset of some $\mathbb{R}^n$. Thus the De Rham approach applies in considerable generality. De Rham cohomology may also be defined directly for a smooth n-manifold, without first choosing some embedding in an euclidean space. We illustrate this for the circle $S^1 = \mathbb{R}/\mathbb{Z}$. (Note that as S^1 is compact it is not diffeomorphic to an open subset of any $\mathbb{R}^n$.)

Example. Let $S = \{f : \mathbb{R} \to \mathbb{R} \mid f \textit{ is } C^\infty, f(x + 1) = f(x) \forall x \in \mathbb{R}\}$ be the set of periodic C^∞ functions. Let $\Omega = Sdx$ and let $d : S \to \Omega$ be given by $df = f'dx$. Then dx is closed but not exact. $\int_{S^1} dx = 1 \neq 0$.

Sometimes we can get away with much "smaller" complexes. The circle is given by polynomial equations: $S^1 = \{(x, y) \in \mathbb{R}^2 \mid x^2 + y^2 = 1\}$. Let $S = \mathbb{R}[x, y]/(x^2 + y^2 - 1) = \mathbb{R}[x][y]/(y^2 = 1 - x^2)$. Since $x^2 + y^2 = 1$ we should have $2xdx + 2ydy = 0$, and so we set $\Omega = (Sdx \oplus Sdy)/S(xdx + ydy) \cong Sdx \oplus \mathbb{R}[x]dy$. Then $dx \wedge dy = (x^2 + y^2)dx \wedge dy = 0$ (on using $xdx = -ydy$, etc.). Hence $\Omega \wedge \Omega = 0$. We find that $\mathrm{Ker}(d) = \mathbb{R}.1$ and $\Omega/dS \cong \mathbb{R}.[xdy]$. Thus $H^q_{DR}(S^1) \cong \mathbb{R}$ for $q = 0, 1$ and is 0 otherwise.

When $n = 3$ we may relate exterior derivation of forms to the familiar operations of vector calculus. Let **i**, **j** and **k** denote the standard unit vectors in $\mathbb{R}^3$. A C^∞ vector field on X is a vector-valued function $f\mathbf{i}+g\mathbf{j}+h\mathbf{k}$ with coefficients in S. Let $\mathcal{V}$ denote the S-module of all such vector fields. Then $\mathcal{V}$ is a free S-module of rank 3, with basis $\{\mathbf{i}, \mathbf{j}, \mathbf{k}\}$. Since $\binom{3}{2} = 3$ and $\binom{3}{3} = 1$ the spaces of 1-forms and 2-forms on $\mathbb{R}^3$ are each free of rank 3, while the space of 3-forms is free of rank 1. There are obvious isomorphisms from $\Omega^1(X)$ and $\Omega^2(X)$ to $\mathcal{V}$, sending dx, dy, dz and $dy \wedge dz, dz \wedge dx, dx \wedge dy$ to **i**, **j**, **k** (respectively), and $\Omega^3(X) \cong S$ via $fdx \wedge dy \wedge dz \mapsto f$. Under these identifications wedge product corresponds to cross product of vector fields and the exterior derivative in degrees 0, 1 and 2 corresponds to *grad*, *curl* and *div*.

If X is a simply connected region of $\mathbb{R}^3$ then $H^1_{DR}(X) = 0$, so $curl(V) = 0$ if and only if $V = grad(f)$ for some function f.

If $H^2_{DR}(X) = 0$ then $div(W) = 0$ if and only if $W = curl(V)$ for some V.

Example. $X = \mathbb{R}^3 \setminus \{O\}$. $V(x) = -(m/|x|^3)x$. $V = d(m/|x|)$.

(Remark. Vector fields and 1-forms are *dual* objects: a 1-form ω may be evaluated on a vector field v to get a function $\omega(v)$. In a proper development of differential geometry a Riemannian metric is used to determine an isomorphism between these modules. Changing the metric changes the isomorphism. The identification of vector fields with 1-forms for open subsets of $\mathbb{R}^3$ given above is the one determined by the standard euclidean inner product on $\mathbb{R}^3$.)

30. Appendix: categories, functors, natural transformations – the bare bones

It has been said that "categories are what one must define in order to define functors, and functors are what one must define in order to define natural transformations" [Freyd, *Abelian Categories*]. For our purposes Category Theory is a language in which the ideas of Algebraic Topology find their most convenient expresssion, and indeed the theory was largely founded by algebraic topologists.

Definition. A *category* $\mathcal{C}$ consists of a class of *objects* and for each ordered pair of objects (A, B) a set $Hom_{\mathcal{C}}(A, B)$ of *morphisms* (or *arrows*) with *domain* (or *source*) A and *codomain* (or *target*) B, written $f : A \to B$, satisfying the following axioms:

(1) for each $f : A \to B$ and $g : B \to C$ there is an unique composite $gf : A \to C$;
(2) if $f : A \to B$, $g : B \to C$ and $h : C \to D$ then $h(gf) = (hg)f$;
(3) for every object A there is an (*identity*) morphism $1_A : A \to A$ such that $f1_A = f$ for all $f : A \to B$ and $1_A g = g$ for all $g : C \to A$.

There is a 1-1 correspondance between objects A and identity morphisms 1_A.

Examples.

(1) $((Set))$ -objects sets, morphisms functions.
(2) $((Top))$ - objects topological spaces, morphisms continuous maps.
(3) $((Grp))$ - objects groups, morphisms group homomorphisms.
(4) $((Mod_R))$ - objects R-modules (for a given ring R), morphisms R-module homomorphisms.
(5) $((Hot))$ - objects topological spaces, morphisms homotopy classes of continuous maps.
(6) $((Top_*))$ - objects topological spaces with basepoints, morphisms continuous, basepoint preserving maps.
(7) $((Hot_*))$ - objects topological spaces with basepoints, morphisms homotopy classes of continuous, basepoint preserving maps (the homotopies must also preserve the basepoint).
(8) if $\mathcal{C}$ is a category the *dual* category $\mathcal{C}^{op}$ has the same class of objects but all the arrows are reversed, so $Hom_{\mathcal{C}^{op}}(A, B) = Hom_{\mathcal{C}}(B, A)$.

Definition. A *covariant* (respectively, *contravariant*) *functor* F from one category $\mathcal{A}$ to another $\mathcal{B}$ is a rule which associates with each object A of $\mathcal{A}$ an object B of $\mathcal{B}$ and with each morphism $f : A_1 \to A_2$ a morphism $F(f) : F(A_1) \to F(A_2)$ (respectively, $F(f) : F(A_2) \to F(A_1)$) such that $F(1_A) = 1_{F(A)}$ and $F(fg) = F(f)F(g)$ (respectively, $F(fg) = F(g)F(f)$), for all objects A and morphisms f, g in $\mathcal{A}$. **Examples.**

(1) The identity functor $I_{\mathcal{A}}$ (or I, for short) is defined by $I(A) = A$ and $I(f) = f$.
(2) There are natural functors from $((Top))$, $((Grp))$, $((Mod_R))$ to $((Set))$ obtained by *forgetting structure* (e.g., every continuous map of topological spaces is a function between the underlying sets. However there is NO such

functor from $((Hot))$ to $((Set))$; in other words, in $((Hot))$ the morphisms are *not* functions.

(3) The abelianization functor $-^{ab}$ from $((Grp))$ to $((Mod_{\mathbb{Z}}))$ which sends a group G the the abelian group $G^{ab} = G/G'$ and a homomorphism $f : G \to H$ to the induced homomorphism $f^{ab} : G/G' \to H/H'$.

(4) If A is an object of $\mathcal{A}$ there is a covariant functor from $\mathcal{A}$ to $((Set))$ sending an object B to the set $Hom_{\mathcal{A}}(A, B)$ and a morphism $f : B \to C$ to $Hom_{\mathcal{A}}(A, f)$, or f_* for short, such that $f_*(g) = fg$ for all $g \in Hom_{\mathcal{A}}(A, B)$. Similarly there is a contravariant functor sending B to $Hom_{\mathcal{A}}(B, A)$ and f to f^* such that $f^*(g) = gf$.

(5) Every contravariant functor from $\mathcal{C}$ to $\mathcal{D}$ can be thought of as a covariant functor from the dual category $\mathcal{C}^{op}$ to $\mathcal{D}$.

Definition. A *natural transformation* η between functors F and G from $\mathcal{A}$ to $\mathcal{Z}$ is a rule which associates to each object A of $\mathcal{A}$ a morphism $\eta(A) : F(A) \to G(A)$ in $\mathcal{Z}$ such that for all morphisms $f : A \to B$ in $\mathcal{A}$ we have

$$\eta(B)F(f) = G(f)\eta(A).$$

Example. Let $\mathcal{V} = ((Mod_{\mathbb{R}}))$ be the category of finite dimensional real vector spaces and D be the contravariant functor from $\mathcal{V}$ to itself which sends a vector space V to its dual $D(V) = Hom_{\mathbb{R}}(V, \mathbb{R})$ (considered as a vector space in the usual way). Then DD is naturally isomorphic to the identity functor, i.e., there is a natural transformation η from $I_{\mathcal{V}}$ to DD such that $\eta(V)$ is an isomorphism for all objects V of $\mathcal{V}$. (It is defined by $\eta(V)(v)(f) = f(v)$ for all $v \in V$ and $f \in D(V)$.)

There is no such *natural* transformation from $I_{\mathcal{V}}$ to D, although each vector space has the same dimension as its dual and so is isomorphic to it.

School of Mathematics and Statistics, University of Sydney, NSW 2006, Australia
E-mail address: jonathan.hillman@sydney.edu.au