THE GENERAL QUADRUPLE POINT FORMULA

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ABSTRACT. Maps between manifolds $M^m \to N^{m+\ell}$ ($\ell > 0$) have multiple points, and more generally, multisingularities. The closure of the set of points where the map has a particular multisingularity is called the multisingularity locus. There are universal relations among the cohomology classes represented by multisingularity loci, and the characteristic classes of the manifolds. These relations include the celebrated Thom polynomials of monosingularities. For multisingularities, however, only the form of these relations is clear in general (due to Kazarian [22]), the concrete polynomials occurring in the relations are much less known. In the present paper we prove the first general such relation outside the region of Morin-maps: the general quadruple point formula. We apply this formula in enumerative geometry by computing the number of 4-secant linear spaces to smooth projective varieties. Some other multisingularity formulas are also studied, namely 5, 6, 7 tuple point formulas, and one corresponding to $\Sigma^2\Sigma^0$ multisingularities.

1. Introduction

Let $f : M^m \to N^n$ be a holomorphic map between compact complex manifolds, and let $\ell = n - m > 0$. Associated with a list of singularities $\underline{\alpha} = (\alpha_1, \dots, \alpha_r)$ one can consider the following multisingularity locus in the source manifold M: the collection of points x where the map has singularity α_1, and $f(x)$ has another $r-1$ preimages $\{x_2, x_3, ..., x_r\}$, with f having singularities α_i at x_i. We will be concerned with the cohomology class $m_{\underline{\alpha}} \in H^*(M)$ represented by the closure of the multisingularity locus, and its image $n_{\underline{\alpha}} \in H^*(N) = H^*(N;\mathbb{Q})$ under the Gysin homomorphism. In the whole paper cohomology is meant with rational coefficients.

The cohomology classes $m_{\underline{\alpha}}$, $n_{\underline{\alpha}}$ satisfy universal identities. The word "universal" means that the dependence of these identities on the manifolds M, N, and the map f, is only via the characteristic classes $c(TM)$, and $f^*(c(TN))$. We mention two such prototype formulas. First, $m_{A_1} = c_{\ell+1}(f)$, that is the cohomology class represented by the points where the map f is singular is equal to the $\ell+1$'st Chern class of the map, where $c(f) = c(f^*TM - TN)$. Another one is $m_{A_0^2} = f^*(n_{A_0}) - c_\ell(f)$. This identity expresses the cohomology class represented by the double point locus, in terms of the cohomology class n_{A_0} represented by the image of f, and the ℓth Chern class of f (we will define precisely what the singularities A_1 and A_0^2 are in Chapter 2).

The universal identities among multisingularity classes have applications in differential topology and algebraic geometry. In differential topology these identities can be used to show that a certain multisingularity locus is not zero, provided only that certain characteristic classes are nonzero. Hence certain degenerations of the map are *forced* by the global topology of the source and target spaces. The polynomials expressing monosingularity classes (called Thom polynomials) are in close relation with polynomials useful in geometry: e.g. relations in presentations of

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cohomology rings of moduli spaces [10], or polynomials governing the combinatorics of Schubert calculus [12], [13].

In the second half of the 20th century the main application of multisingularity class identities were in enumerative geometry. When we want to count certain geometric objects satisfying certain properties, we can often encode the problem by setting up a map, whose certain multisingularity locus is in bijection with the counted objects. Characteristic classes of maps are usually easy to handle, so using the universal identities, we have a formula for the counted objects. This approach was used by e.g. Kleiman [23], Katz [19], Colley [6], and recently by Kazarian [20], [22] with great success. One advantage of this method is that it often avoids the problem of excess intersection.

The limitation of this method is the fact that hardly any *general* multisingularity identities are known. By "general" we mean a formula which is valid for all dimensional settings, and for all maps with expected multisingularities. For example the two prototype formulas above are valid for any dimension $m < n$, but e.g. $m_{A_2} = c_1^2(f) + c_2(f)$ is only valid for maps with $\ell = 0$. Some 'general' monosingularity formulas are known, and only three general multisingularity formulas. These are the double-point formula, the triple-point formula, and a formula concerning A_0A_1 multisingularities. The reason why higher formulas are considerably harder is roughly speaking that higher multisingularities do not only interfere with the simplest monosingularities of maps, the so-called Morin singularities (a.k.a. corank-one, or curvilinear singularities). The quadruple points of a map $M^m \to N^{m+\ell}$ have codimension 3ℓ in the source, while the codimension of non-Morin singularities is $2\ell + 4$. Hence the non-Morin ones interfere with the quadruple points.

Methods of algebraic geometry have been applied to find several multiple point formulas that are valid for maps with only Morin singularities, see works of Kleiman, Kazarian [24], [20] and references therein. These formulas are not general either, because they agree with the general formulas only up to characteristic classes supported on non-Morin singularity loci.

The main result of the present paper is proving the general i-tuple point formula (20) for $i \leq 7$. Our method has three pillars, as follows: (i) Kazarian found the general form of multisingularity formulas [22]; (ii) Rimányi—based on Szűcs's construction of classifying space of multisingularities in [33]—found an "interpolation" method to gather information on the polynomials governing the multisigularity identities [31]; (iii) recently Bérczi and Szenes in [5] used advanced localizations to calculate the Thom polynomial of the A_i singularity for $i \leq 6$ (the A_3 formula was announced in [4]). What we will show using interpolation is that after certain identifications, the residue polynomial of e.g. quadruple points is equal to the Thom polynomial of A_3 singularities—for a different dimension setting.

Since the interpolation method is topological in nature, our method is topological. In Section 6 we will show an enumerative geometry application. Namely, we will calculate the number (the cohomology class) of 4-secant linear spaces to a smooth projective variety. For a smooth surface in 10 dimensional projective space we recover the Hilbert scheme calculation of [26], for the other dimension settings our result is new.

The advance of this paper, that is, the step from triple point formulas to higher multiple point formulas can be compared with a recent advance in algebraic geometry: from the study of the space of triangles [7] to the study of the space of tetrahedra [3].

Our approach to reduce multisingularity polynomials to some other, easier and known ones has two limitations. First, not many general Thom polynomials (called Thom series) are known;

the best results seem to be [5], [11]. The other limitation is that the direct interpolation method breaks down where moduli of singularities occur, that is in codim $\geq 6\ell + 9$.

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2. Singularities and multisingularities

2.1. **Contact singularities of maps.** Fix integers $m < n$ and let $\ell = n - m$. We say that maps or map germs mapping from an m dimensional space to an n dimensional space have *relative dimension* ℓ.

Consider $\mathcal{E}(m,n)$, the vector space of holomorphic map germs $(\mathbb{C}^m, 0) \to \mathbb{C}^n$. The subspace consisting of germs $(\mathbb{C}^m, 0) \to (\mathbb{C}^n, 0)$ is denoted by $\mathcal{E}^0(m,n)$. The vector space $\mathcal{E}(m) := \mathcal{E}(m,1)$ is a local algebra with maximal ideal $\mathcal{E}^0(m)$. The space $\mathcal{E}(m,n)$ is a module over $\mathcal{E}(m)$, with $\mathcal{E}^0(m,n)$ a submodule. A map $f \in \mathcal{E}^0(m,n)$ induces a pullback $f^* : \mathcal{E}(n) \to \mathcal{E}(m)$ by composition.

Definition 2.1. *The local algebra Q_f of a germ $f \in \mathcal{E}^0(m,n)$ is defined by $Q_f = \mathcal{E}(m)/(f^* \mathcal{E}^0(n))$.*

We will be concerned with germs f for which the local algebra is finite dimensional; i.e. the ideal $(f^* \mathcal{E}^0(n))$ contains a power of the maximal ideal. We call these germs *finite*. For such a germ, in local coordinates, $f = (f_1(x_1, \ldots, x_m), \cdots, f_n(x_1, \ldots, x_m))$, we have $Q_f = \mathbb{C}[[x_1, \ldots, x_m]]/(f_1, \ldots, f_n)$.

Definition 2.2. *For two germs, f and g, we say that f is* contact equivalent *to g, if $Q_f \cong Q_g$; that is, their local algebras are isomorphic. An equivalence class $\eta \subset \mathcal{E}^0(m,n)$ will be called a (contact) singularity.*

In singularity theory one considers the so called contact group $\mathcal{K}(m,n)$ acting on the vector space $\mathcal{E}^0(m,n)$, and defines germs to be contact equivalent if they are in the same orbit. It is a theorem of Mather [27] that for finite germs the two definitions are equivalent. Thus, for the rest of this paper all singularities to which we refer will be finite in the previous sense.

Remark 2.3. The group $\mathcal{K}(m,n)$ contains the group of holomorphic reparametrizations of the source $(\mathbb{C}^m, 0)$ and target spaces $(\mathbb{C}^n, 0)$. Hence for a map $f : M^m \to N^n$ between manifolds it makes sense to talk about the contact singularity of f at a point in M. Hence, for a map $f : M^m \to N^n$ and a singularity $\eta \subset \mathcal{E}^0(m,n)$, we can define the singularity subset

$$\eta(f) = \{x \in M | \text{ the germ of } f \text{ at } x \text{ belongs to } \eta\}.$$

2.2. **The zoo of singularities.** The classification of finite singularities is roughly the same as the classification of finite dimensional commutative local $\mathbb{C}$-algebras. Only 'roughly', because for a given m and n only algebras that can be presented by m generators and n relations turn up as local algebras of singularities $\mathbb{C}^m, 0 \to \mathbb{C}^n, 0$.

A natural approach is to try to classify singularities in the order of their codimensions in $\mathcal{E}^0(m,n)$. For large ℓ the classification of small codimensional singularities is as follows (see e.g. [2]).

codim	0	$\ell+1$	$2\ell+2$	$2\ell+4$	$3\ell+3$	$3\ell+4$	$3\ell+5$
Σ^0	A_0						
Σ^1		A_1	A_2		A_3		
Σ^2				$III_{2,2}$		$I_{2,2}$	$III_{2,3}$

Here we use the following notations: A_i means the singularity with local algebra $\mathbb{C}[x]/(x^{i+1})$; $I_{a,b}$ means the singularity with local algebra $\mathbb{C}[x,y]/(xy, x^a+y^b)$; and $III_{a,b}$ means the singularity with local algebra $\mathbb{C}[x,y]/(x^a, xy, y^b)$. The symbol Σ^r is a property of a singularity, it means that the derivative drops rank by r, equivalently, that the local algebra can be minimally generated by r generators. The $\Sigma^{\leq 1}$ singularities are called Morin singularities (a.k.a. corank 1, or curvilinear singularities). As one studies singularities of high codimension, they appear in moduli. However for the main result of the present paper we can avoid working with them.

Observe that we gave the classification independent of m and n, that is, we gave the same name for singularities for different dimension settings. E.g. the following are all A_2 germs: $x \mapsto x^3$ ($m = n = 1$), $(x, y) \mapsto (x^3, y)$ ($m = 2, n = 2$), $(x, y) \mapsto (x^3 + xy, y)$ ($m = n = 2$). An essential difference between the latter two is that the last one is *stable* (it is called cusp singularity), the other one is not (stable representatives will play an important role in Section 3.1).

As we already noted in Remark 2.3, singularity submanifolds can stratify the source space of a map $f : M^m \to N^n$ between manifolds. We want to study a finer stratification though—one which corresponds to multisingularities.

2.3. **Multisingularities.**

Consider contact singularities $\alpha_i \subset \mathcal{E}^0(m, n)$, $m < n$.

Definition 2.4. *A* multisingularity $\underline{\alpha}$ *is a multi-set of singularities* $(\alpha_1, \ldots, \alpha_r)$ *together with a distinguished element, denoted* α_1*.*

For reasons explained in Section 2.4, we define the codimension of a multisingularity $(\alpha_1, \ldots, \alpha_r)$ by

$$\operatorname{codim} \underline{\alpha} = (r-1)\ell + \sum \operatorname{codim} \alpha_i. \tag{1}$$

Hence the codimension does not depend on the order of the monosingularities. The list of multisingularities of small codimension (when ℓ is large) is given in the following table.

codim	0	ℓ	$\ell+1$	2ℓ	$2\ell+1$	$2\ell+2$	$2\ell+4$	3ℓ	$3\ell+1$	$3\ell+2$	$3\ell+3$	$3\ell+4$
Σ^0	A_0	A_0^2		A_0^3				A_0^4				
Σ^1			A_1		A_1A_0	A_2			$A_1A_0^2$	A_2A_0 A_1^2	A_3	
Σ^2							$III_{2,2}$					$III_{2,2}A_0$ $I_{2,2}$

Here we used the notation $\alpha_1\alpha_2\ldots$ for the multiset $(\alpha_1, \alpha_2, \ldots)$, and any of its permutations.

Definition 2.5. *Let* $f : M^m \to N^n$ *be a holomorphic map of complex manifolds, and* $\underline{\alpha} = (\alpha_1, \alpha_2, \ldots, \alpha_r)$ *a multisingularity. We define the following multisingularity loci in* M *and* N

$$M_{\underline{\alpha}} = \{x_1 \in M | f(x_1) \text{ has exactly } r \text{ pre-images } x_1, \ldots, x_r, \text{ and } f \text{ has singularity } \alpha_i \text{ at } x_i\},$$

and $N_{\underline{\alpha}} = f(M_{\underline{\alpha}})$*.*

If we permute the monosingularities in $\underline{\alpha}$, i.e. choose another singularity to be α_1, then $M_{\underline{\alpha}}$ changes, while $N_{\underline{\alpha}}$ does not.

2.4. **Admissible maps.** The main point of the present paper is to study certain identities among cohomology classes represented by multisingularity loci. We can only expect such identities if the map satisfies certain transversality conditions. We will define these maps in the present section, and call them 'admissible'.

As before, the codimension of the singularity η in $\mathcal{E}^0(m,n)$ is denoted by $\operatorname{codim}\eta$. It is reasonable to expect that for a 'nice enough' map $f : M \to N$, the codimension of $\eta(f)$ in M is the same (see Remark 2.3). Indeed, for a map $f : M^m \to N^n$ one can consider the bundle

$$\{\text{germs } (M,x) \to (N, f(x))\} \to \{(x, f(x)) | x \in M\}$$

together with the section $(x, f(x)) \mapsto$ the germ of f at x. (Precisely speaking, one should consider jet approximations to have a finite rank bundle.) The fibers of this bundle are identified with $\mathcal{E}^0(m,n)$, so we can consider η in each. Thus we obtain a submanifold of codimension $\operatorname{codim}\eta$ in the total space. The set $\eta(f)$ is the preimage of this submanifold along the section. Hence, for transversal sections the codimension of $\eta(f)$ in M is $\operatorname{codim}\eta$. An 'admissible-for-monosingularities' map must have this transversality property. If we worked over the real numbers we would have the transversality theorem guaranteeing that almost all maps are admissible-for-monosingularities.

We need, however, the admissibility property for multisingularities as well. This more sophisticated notion uses the classifying space of multisingularities, see [33], [22] as follows. A map $f : M \to N$ induces a map k_f from N to a space X called the *classifying space of multisingularities*. The infinite dimensional space X has a finite codimensional submanifold $X_{\underline{\alpha}}$ corresponding to the multisingularity $\underline{\alpha}$. The set $N_{\underline{\alpha}}$ is the preimage of $X_{\underline{\alpha}}$ along the map k_f. The map f is defined to be admissible if k_f is transversal to $X_{\underline{\alpha}}$ for all $\underline{\alpha}$. Over the real numbers almost all maps are admissible.

Remark 2.6. Another way to define admissibility is to require that a natural section of the "multijet bundle" is transversal to certain submanifolds in the total space, as in the Multijet Transversality Theorem, see [18, Thm. 4.13]. Either way, admissible maps are admissible-for-monosingularities, together with the property that the closures of the f-images of the monosingularity submanifolds $\alpha(f)$ satisfy certain transversality properties. In Figure 1 both maps are admissible-for-monosingularities, but only the first map is admissible for multisingularities.

The codimension of $X_{\underline{\alpha}}$ in X is $r\ell + \sum \operatorname{codim}\alpha_i$. Therefore, for an admissible map, the codimension of $N_{\underline{\alpha}}$ in N is $r\ell + \sum \operatorname{codim}\alpha_i$; and the codimension of $M_{\underline{\alpha}}$ in M is $(r-1)\ell + \sum \operatorname{codim}\alpha_i$, (*cf.* formula (1)). It also follows that for admissible maps, the closures of the loci $M_{\underline{\alpha}}$ and $N_{\underline{\alpha}}$ support fundamental homology classes. We will call the Poincaré duals of these classes the cohomology classes represented by the $\underline{\alpha}$ multisingularity loci in the source and target manifolds.

2.5. **Cohomology classes represented by multisingularity submanifolds.** Let $f : M^m \to N^n$ be an admissible map. Denote by $\overline{m}_{\underline{\alpha}} = \left[\bar{M}_{\underline{\alpha}}\right] \in H^{\operatorname{codim}\underline{\alpha}}(M)$ the cohomology class represented by the closure of the α-multisingularity locus in the source, and $\overline{n}_{\underline{\alpha}} = \left[\bar{N}_{\underline{\alpha}}\right] \in H^{\ell+\operatorname{codim}\underline{\alpha}}(N)$, in the target. Since it is often of use to consider these classes $\overline{m}_{\underline{\alpha}}$, $\overline{n}_{\underline{\alpha}}$ with their natural multiplicities, we let

$$m_{\underline{\alpha}} = \#Aut(\alpha_2, \ldots, \alpha_r)\overline{m}_{\underline{\alpha}}, \qquad n_{\underline{\alpha}} = \#Aut(\alpha_1, \alpha_2, \ldots, \alpha_r)\overline{n}_{\underline{\alpha}},$$

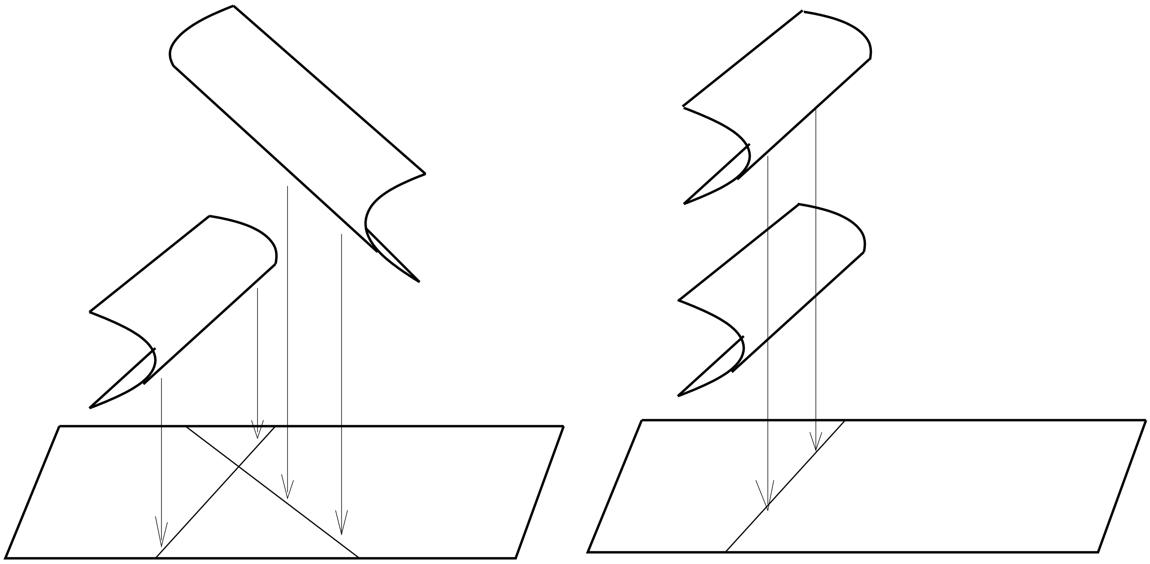

FIGURE 1

where $\#Aut(\alpha_1, \alpha_2, \ldots, \alpha_r) = \#Aut(\underline{\alpha})$ is the number of permutations $\sigma \in \mathfrak{S}_r$ such that $\alpha_{\sigma(i)} = \alpha_i$ for all i from 1 to r. So if $\underline{\alpha}$ contains k_1 singularities of type α_1, k_2 of type α_2, etc., then $\#Aut(\underline{\alpha}) = k_1!k_2!\ldots$.

The degree of the restriction map $f : M_{\underline{\alpha}} \to N_{\underline{\alpha}}$ is the number of α_1 singularities in $\underline{\alpha}$, hence we have the following relation

$$n_{\underline{\alpha}} = f_!(m_{\underline{\alpha}}), \tag{2}$$

where $f_!$ is the Gysin homomorphism.

2.6. **Classes of multisingularity loci in terms of characteristic classes.** The virtual normal bundle $\nu(f)$ of a map $f : M^m \to N^n$ is the formal difference $f^*(TN) - TM$ of bundles over M. This is an actual bundle if f is an immersion. The total Chern class of a map is defined to be the total Chern class of its virtual normal bundle,

$$c(f) = c(f^*(TN) - TM) = \frac{c(f^*(TN))}{c(TM)} = \frac{f^*(c(N))}{c(M)}.$$

A classical theorem of Thom [36] is that *mono*singularity loci in the source can be expressed as a polynomial (the Thom polynomial) of the Chern classes of the map. The generalization for multisingularity loci was found by Kazarian.

Theorem 2.7 (Kazarian [22])**.** *For multisingularities $\underline{\alpha} = (\alpha_1, \ldots, \alpha_r)$ and $J \subset \{1, \ldots, r\}$, let $\bar{J} = \{1, \ldots, r\} \setminus J$. There exist unique polynomials $R_{\underline{\alpha}}$ in the Chern classes of the virtual normal bundle $\nu(f)$, called* residue (or residual) polynomials*, satisfying*

$$m_{\underline{\alpha}} = R_{\underline{\alpha}} + \sum_{1 \in J \subsetneq \{1,\ldots,r\}} R_{\underline{\alpha}_J} f^*(n_{\underline{\alpha}_{\bar{J}}}) \tag{3}$$

for admissible maps. Here the sum is taken over all possible subsets of $\{1, \ldots, r\}$ containing 1. Moreover the residue polynomials are independent of the order of the monosingularities α_i in $\underline{\alpha}$.

In particular, if $\underline{\alpha} = (\alpha)$ is a monosingularity, then (3) yields $m_\alpha = R_\alpha$, hence R_α is the Thom polynomial Tp_α of the given singularity. For example, for $n = m$ we have $R_{A_2} = c_1^2 + c_2$; and this means that the cohomology class represented by points in M where the map has singularity A_2, is equal to $c_1^2 + c_2$ of the virtual normal bundle of the map.

Observe that in the last example we did not specify m and n, only their difference. This is a classical fact about Thom polynomials: the Thom polynomial of singularities having the same local algebra and the same relative dimension ℓ (but maybe living in different vector spaces $\mathcal{E}^0(m,n)$) are the same.

We can set $S_{\underline{\alpha}} = f_!(R_{\underline{\alpha}})$, and putting (3) together with (2) and the adjunction formula for the Gysin map yields

$$n_{\underline{\alpha}} = S_{\underline{\alpha}} + \sum_{1\in J\subsetneq\{1,\dots,r\}} S_{\underline{\alpha}_J} n_{\underline{\alpha}_{\overline{J}}}. \tag{4}$$

3. Calculation of $R_{\underline{\alpha}}$

Different calculational techniques for residue polynomials of monosingularities, that is, Thom polynomials of contact singularities has been studied for decades. One of the most effective techniques, which also generalizes to residue polynomials of multisingularities was invented by the second author. We will call it the interpolation method, and summarize it below. For more details and proofs see [30], [14]

3.1. **Interpolation.** Let ξ be a contact singularity, and let us choose a *stable* representative $\xi' \in \mathcal{E}^0(m,n)$. Stability of germs is discussed e.g. in [2]. In later sections we will not distinguish ξ from ξ', and call both by the same name ξ.

One of the main ideas of [30] is—roughly speaking—that we can pretend that ξ' is a map. Then stability of the germ implies that as a map it is admissible, hence formulas (3) and (4) hold for it. However, the source and target spaces of ξ' are (germs of) vector spaces, their cohomology ring is trivial, so the formulas are meaningless. The idea is that we consider formulas (3) and (4) for ξ' in equivariant cohomology. For this we need a group action.

A pair (ϕ,ψ) is a symmetry of the germ ξ', if ϕ (resp. ψ) is an invertible element in $\mathcal{E}^0(m,m)$ (resp. $\mathcal{E}^0(n,n)$), and

$$\xi' = \psi \circ \xi' \circ \phi^{-1}.$$

If G is a group of symmetries of ξ', then all ingredients of formulas (3) and (4) make sense in the G-equivariant cohomology ring of $\mathbb{C}^m$ (resp. $\mathbb{C}^n$). The equivariant cohomology of a vector space is the same as the equivariant cohomology of the one point space, i.e. the ring of the G-characteristic classes H^*BG. Observe also, that the map $\xi'^* : H^*_G(\mathbb{C}^n) \to H^*_G(\mathbb{C}^m)$ is the identity map of H^*BG.

The fact that formulas (3) and (4) hold for stable singularities in equivariant cohomology put strong constraints on the residue polynomials.

Example 3.1. Consider the stable germ $\xi : (x,y) \mapsto (x^2, xy, y)$, called Whitney umbrella. The group $G = U(1) \times U(1)$ is a group of symmetries of ξ with the representations

$$(\alpha,\beta)\cdot(x,y) = (\alpha x, \beta\bar{\alpha}y), \qquad (\alpha,\beta)\cdot(u,v,w) = (\alpha^2 u, \beta v, \beta\bar{\alpha}w), \qquad \big((\alpha,\beta)\in U(1)\times U(1)\big)$$

on the source and target spaces respectively. Indeed,

$$\xi(\ (\alpha,\beta)\cdot(x,y)\ ) = (\alpha,\beta)\cdot\xi(x,y).$$

We also use α and β for the first Chern classes of the two factors of $U(1)\times U(1)$. Then $H^*BG = \mathbb{Q}[\alpha,\beta]$, and we have that

$$c(\xi) = \frac{(1+2\alpha)(1+\beta)(1+\beta-\alpha)}{(1+\alpha)(1+\beta-\alpha)} = 1 + (\beta+\alpha) + (\alpha\beta-\alpha^2) + (-\alpha^2\beta+\alpha^3) + \dots.$$

The closure of the double point locus (in the source space) of the map ξ is $\{y=0\}$. Its cohomology class is therefore $\beta-\alpha$, the equivariant Euler class of its normal bundle. The cohomology class represented by the image of this map can be calculated to be 2β (see lemma 5.2 below). The pullback map ξ^* is an isomorphism (as for all germs), hence the pullback of the cohomology class of the image of ξ is 2β. One of Kazarian's formulas (3) (for maps from 2 dimensions to 3 dimensions) states that the difference of these two multisingularity classes is $R_{A_0^2}$. Hence we get that

$$(\beta-\alpha) - 2\beta = R_{A_0^2}(c_1 = \beta+\alpha, c_2 = \alpha\beta-\alpha^2, \dots) \in \mathbb{Q}[a,b].$$

This has only one solution for $R_{A_0^2}$, namely $R_{A_0^2} = -c_1$.

Conditions obtained from stable singularities often determine uniquely the residue polynomials, as follows. Let α be a multisingularity of codimension d, and suppose that there are only finitely many monosingularities ξ with codimension $\le d$. For each ξ we can consider the maximal compact symmetry group G_ξ (see [32]) of a stable representative. It is explained in [15] that G_ξ acts on the normal bundle of $\xi \subset \mathcal{E}^0(m,n)$. Then, we have the following theorem.

Theorem 3.2. [15] *Suppose the G_ξ-equivariant Euler class of the normal bundle of the embedding $\xi \subset \mathcal{E}^0(m,n)$ is not a 0-divisor for all the finitely many singularities ξ with* $\operatorname{codim}\xi \le d$. *If formula (3) holds for stable representatives of all the finitely many ξ with codimension $\le d$ (in G_ξ equivariant cohomology), then formula (3) holds for all admissible maps.*

Strictly speaking this theorem is proved in [15] only for monosingularities (since that was the object of the paper). However, what is proved there, is that the map

$$\mathbb{Q}[c_1, c_2, \dots] \to \oplus H^*(BG_\xi),$$

whose component functions are the evaluations of Chern classes at the stable representatives of the ξ's with codim $\le d$, is injective in degrees $\le d$. This implies the result for multisingularities as well, provided the existence and the uniqueness of residual polynomials stated in Theorem 2.7.

Mather [27] determined the codimensions in which moduli of singularities occur: for large ℓ moduli occurs in codimension $6\ell+9$. Calculations show that the condition in the theorem on the Euler classes of the monosingularities of codimension $\le 6\ell+8$ also hold.

3.2. **A sample Thom polynomial calculation.** We will show how Theorem 3.2 can be used to find the Thom polynomial of A_1 (a classical result, due to Giambelli, Whitney, Thom in various disguises). We will carry out the calculation for general ℓ.

The codimension of the A_1 singularity is $\ell+1$, hence Tp_{A_1} is a degree $\ell+1$ polynomial, such that

$$[\overline{A_1(f)}] = \mathrm{Tp}_{A_1}(c(f)) \tag{5}$$

for any admissible map f. There are only two singularities with codimension $\le \ell+1$, namely: A_0 and A_1. Hence from Theorem 3.2 we can deduce two constraints on the Tp_{A_1}. It turns out that the constraint coming from A_0 is redundant, hence we will now consider the constraint coming from A_1 itself. For this we need to choose a stable representative of the singularity A_1. The general procedure of finding a stable representative of a singularity given by its local algebra is called "universal unfolding". For A_1 we obtain the following germ $\mathbb{C}^{\ell+1},0 \to \mathbb{C}^{2\ell+1},0$:

$$f:(x,y_1,\dots,y_\ell)\mapsto(x^2,xy_1,\dots,xy_\ell,y_1,\dots,y_\ell).$$

The general procedure to find the maximal compact symmetry group is described in [32]. For our germ we obtain $G_f=U(1)\times U(\ell)$ with the representations

$$\rho_1\oplus(\overline{\rho_1}\otimes\rho_\ell),\qquad \rho_1^2\oplus\rho_\ell\oplus(\overline{\rho_1}\otimes\rho_\ell)$$

on the source and target spaces, where ρ_1 and ρ_ℓ are the standard representations of $U(1)$ and $U(\ell)$. It is easier to understand the representations of the maximal torus $U(1)\times U(1)^\ell$, so we proceed as follows. Let $(\alpha,\beta_1,\dots,\beta_\ell)\in U(1)\times U(1)^\ell$. The diagonal actions given by

$$(\alpha,\bar\alpha\beta_1,\dots,\bar\alpha\beta_\ell),\qquad\text{and}\qquad(\alpha^2,\beta_1,\dots,\beta_\ell,\bar\alpha\beta_1,\dots,\bar\alpha\beta_\ell)$$

is clearly a symmetry of the germ above.

Hence, when we apply formula (5) to the germ f, we obtain an equation in $H^*(B(U(1)\times U(\ell)))$. By abuse of language we denote the Chern roots of $U(1)$ and $U(\ell)$ by α and $\beta_1,\dots,\beta_\ell$. Let b_i be the i'th elementary symmetric polynomial of the β_i's, that is the universal Chern classes of the group $U(\ell)$. Then the total Chern class of f is

$$c(f)=\frac{(1+2\alpha)\prod^\ell(1+\beta_i)\prod^\ell(1+\beta_i-\alpha)}{(1+\alpha)\prod^\ell(1+\beta_i-\alpha)}=\frac{(1+2\alpha)\prod^\ell(1+\beta_i)}{(1+\alpha)}=$$
$$=1+(b_1+\alpha)+(b_2+b_1\alpha-\alpha^2)+(b_3+b_2\alpha-b_1\alpha^2+\alpha^3)+\dots,$$

that is, $c_1(f)=b_1+\alpha$, $c_2(f)=b_2+b_1\alpha-\alpha^2$, etc.

Now we need the left hand side of formula (5) for our germ f. The A_1 locus of the germ f is only the origin, hence $[A_1(f)]$ is the class represented by the origin. By definition the class represented by the origin in the equivariant cohomology of a vector space is the Euler class (a.k.a. top Chern class) of the representation. In our case it is

$$\alpha\prod^\ell(\beta_i-\alpha).$$

Hence formula (5) reduces to

$$\alpha\prod^\ell(\beta_i-\alpha)=\mathrm{Tp}_{A_1}(c_1=b_1+\alpha,c_2=b_2+b_1\alpha-\alpha^2,\dots).$$

It is simple algebra to show that the polynomials $b_1+\alpha, b_2+b_1\alpha-\alpha^2,\dots$ (up to the degree $\ell+1$ one) are algebraically independent in $\mathbb{Q}[\alpha,b_1,b_2,\dots,b_\ell]$, and that $c_{\ell+1}=\alpha\prod^\ell(\beta_i-\alpha)$. This yields that $\mathrm{Tp}_{A_1}=c_{\ell+1}$.

Remark 3.3. The ingredients of Kazarian's formulas (3) are certain geometrically defined classes (m_α, n_α), as well as the Chern classes of the map. When applying these formulas for stable representatives of monosingularities, there is an essential simplification concerning only the Chern classes. The stable representatives are universal unfoldings of so-called *genotypes* of the singularity. The fact is that the genotype has the same symmetry group as its universal unfolding; moreover, the Chern classes of the genotype are the same as the Chern classes of its universal unfolding, see [30]. Hence, later in the paper, if we only need the Chern classes of a stable representative, we may work with the genotype instead.

Remark 3.4. The ring of characteristic classes of a group G embeds into the ring of characteristic classes of its maximal torus T. Hence the information that a formula holds in $H^*(BG)$ is the same as that it holds in $H^*(BT)$. In what follows we will always use the one more convenient for our notation.

4. The known general residue polynomials

Infinitely many Thom polynomials can be named at the same time, due to certain stabilization properties that they satisfy. In the present paper we will be concerned with two of the stabilizations. The first we already mentioned, namely that the Thom polynomial only depends on ℓ, not on m and n (for the same local algebra). The second—Theorem 4.1 below—concerns the Thom polynomial as ℓ varies (while not changing the local algebra). To phrase Theorem 4.1 we need some notions.

Let Q be a local algebra of a singularity. In singularity theory one considers three integer invariants of Q as follows: (i) $\delta = \delta(Q)$ is the complex dimension of Q, (ii) the *defect* $d = d(Q)$ of Q is defined to be the minimal value of $b - a$ if Q can be presented with a generators and b relations; (iii) the definition of the third invariant $\gamma(Q)$ is more subtle, see [27, §6]. The existence of a *stable* singularity $(\mathbb{C}^m, 0) \to (\mathbb{C}^n, 0)$ with local algebra Q is equivalent to the conditions $\ell \geq d$, $\ell(\delta - 1) + \gamma \leq m$. Under these conditions the codimension of the contact singularity with local algebra Q in $\mathcal{E}^0(m, n)$ is $\ell(\delta - 1) + \gamma$.

Theorem 4.1. [15] *Let Q be a local algebra of singularities. Assume that the normal Euler classes of the singularities in $\mathcal{E}^0(m, n)$ with local algebra Q are not 0. Then associated with Q there is a formal power series (*Thom series*) Ts_Q in the variables $\{d_i | i \in \mathbb{Z}\}$, of degree $\gamma(Q) - \delta(Q) + 1$, such that all of its terms have $\delta(Q) - 1$ factors, and the Thom polynomial of $\eta \subset \mathcal{E}^0(m, n)$ with local algebra Q is obtained by the substitution $d_i = c_{i+(m-n+1)}$.* □

Even though there are powerful methods by now to compute individual Thom polynomials (i.e. finite initial sums of the Ts), finding closed formulas for these Thom series remains a subtle problem. Here are some examples.

$\mathbf{A}_0$: $Q = \mathbb{C}$ (*embedding*). Here $\delta = 1$, $\gamma = 0$, and
$$\mathrm{Ts} = 1.$$

$\mathbf{A}_1$: $Q = \mathbb{C}[x]/(x^2)$ (e.g. *fold, Whitney umbrella*). Here $\delta = 2$, $\gamma = 1$, and
$$\mathrm{Ts} = d_0.$$

$\mathbf{A}_2$: $Q = \mathbb{C}[x]/(x^3)$ (e.g. *cusp*). Here $\delta = 3$, $\gamma = 2$, and (see [34])
$$\mathrm{Ts} = d_0^2 + d_{-1}d_1 + 2d_{-2}d_2 + 4d_{-3}d_3 + 8d_{-4}d_4 + \dots.$$

$\mathbf{A}_3$: $Q = \mathbb{C}[x]/(x^4)$. Here $\delta = 4$, $\gamma = 3$, and (see [4, Thm.4.2], [5], [25])
$$\mathrm{Ts} = \sum_{i=0}^{\infty} 2^i d_{-i} d_0 d_i + \frac{1}{3} \sum_{i=1}^{\infty} \sum_{j=1}^{\infty} 2^i 3^j d_{-i} d_{-j} d_{i+j} + \frac{1}{2} \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} a_{i,j} d_{-i-j} d_i d_j,$$

where $a_{i,j}$ is defined by the formal power series
$$\sum_{i,j} a_{i,j} u^i v^j = \frac{u\frac{1-u}{1-3u} + v\frac{1-v}{1-3v}}{1 - u - v}.$$

Although we used formal power series to describe Thom polynomials, of course, the Thom polynomials themselves are polynomials, since only finitely many terms survive for any concrete ℓ. For example from the Thom series of A_2 above it follows that for $\ell = 1$ the Thom polynomial is $c_1^2 + c_2$, for $\ell = 2$ the Thom polynomial is $c_2^2 + c_1c_3 + 2c_4$, etc.

There are other Thom series known in *iterated residue form*: Bérczi and Szenes found the Thom series of A_i singularities for $i \leq 6$ [5]. In an upcoming paper [11] the Thom series corresponding to several non-Morin singularities are calculated. In [9] the Thom series of some second order Thom-Boardman singularities are calculated.

However, all the mentioned results are Thom polynomials, that is residue polynomials of monosingularities, rather than multisingularities. Several individual multisingularity residue polynomials are calculated for small ℓ in [22] and [21]. However, the methods used there do not easily extend to find formulas for all ℓ. For example it was known that

$$R_{A_0^4} = -6(c_1^3 + 3c_1c_2 + 2c_3) \qquad \text{for } \ell = 1, \tag{6}$$

$$R_{A_0^4} = -6({c_2}^3 + 3\, c_1c_2c_3 + 7\, c_2c_4 + 2\, {c_1}^2c_4 + 10\, c_1c_5 + 12\, c_6 + {c_3}^2) \qquad \text{for } \ell = 2, \tag{7}$$

but no $R_{A_0^4}$ formula was known for all ℓ. In other words the *residue series*, i.e. a formula containing ℓ as a parameter is known only for a very few multisingularities. Here is a complete list of those:

Theorem 4.2. [35] *For admissible maps $f : M^m \to N^n$ we have*

$$m_{A_0^2} = f^*(n_{A_0}) - c_\ell(f).$$

That is, the residue polynomial of the multisingularity A_0^2 is $-c_\ell$.

Theorem 4.3. [8] *For admissible maps $f : M^m \to N^n$ we have*

$$m_{A_0^3} = f^*(n_{A_0^2}) - 2c_\ell f^*(n_{A_0}) + 2\Big(c_\ell^2 + \sum_{i=0}^{\ell-1} 2^i c_{\ell-1-i}c_{\ell+1+i}\Big).$$

That is, the residue polynomial of the multisingularity A_0^3 is

$$R_{A_0^3} = 2\Big(c_\ell^2 + \sum_{i=0}^{\ell-1} 2^i c_{\ell-1-i}c_{\ell+1+i}\Big).$$

Theorem 4.4. [20] *For admissible maps $f : M^m \to N^n$ we have*

$$m_{A_1A_0} = f^*(n_{A_0}) - 2\Big(c_\ell c_{\ell+1} + \sum_{i=0}^{\ell-1} 2^i c_{\ell-1-i}c_{\ell+2+i}\Big),$$

$$\begin{aligned} m_{A_0A_1} &= f^*(n_{A_1}) - 2\Big(c_\ell c_{\ell+1} + \sum_{i=0}^{\ell-1} 2^i c_{\ell-1-i}c_{\ell+2+i}\Big) \\ &= f^*(f_!(c_{\ell+1})) - 2\Big(c_\ell c_{\ell+1} + \sum_{i=0}^{\ell-1} 2^i c_{\ell-1-i}c_{\ell+2+i}\Big). \end{aligned}$$

That is, the residue polynomial of the multisingularity A_1A_0 is

$$R_{A_0A_1} = -2\Big(c_\ell c_{\ell+1} + \sum_{i=0}^{\ell-1} 2^i c_{\ell-1-i}c_{\ell+2+i}\Big).$$

There are basically two main reasons why the calculation of other residue polynomials is more difficult.

First, no transparent geometric meaning of residue polynomials of multisingularities has been found so far. While residue polynomials of monosingularities are equivariant classes represented by geometrically relevant varieties in $\mathcal{E}^0(m,n)$, hence they are part of equivariant cohomology, the residue polynomials of multisingularities do not seem to be part of equivariant cohomology. In other words, the cohomology ring of the classifying space of singularities is a ring of characteristic classes, while the cohomology ring of the classifying space of multisingularities contains Landweber-Novikov classes (see more details in [22]). Hence powerful techniques of equivariant cohomology (e.g. localization) can not be used directly for multisingularities.

The second reason can be seen in the diagram of multisingularities in Section 2.3. The codimension of the multisingularities considered in the above three theorems are smaller than the codimension of any non-Morin, (i.e. $\Sigma^{\geq 2}$) singularity. Therefore, non-Morin singularities can be disregarded when studying those three multisingularities. As the table shows, we will have "competing" non-Morin singularities for any other multisingularity.

The main result of the present paper is the calculation of residue polynomials in such non-Morin cases, namely the residue polynomial $R_{A_0^i}$ for all ℓ and $i \leq 7$.

5. General quadruple point formula

In order to emphasize the relative dimension, let $R_{\underline{\alpha}}(\ell)$ denote the residue polynomial of the multisingularity $\underline{\alpha}$ for maps of relative dimension ℓ. We are now ready to state the main theorem.

Theorem 5.1. *For $i \leq 6$ we have*

$$R_{A_0^{i+1}}(\ell) = (-1)^i i! R_{A_i}(\ell-1).$$

Since the polynomial R_{A_i} is known for $i \leq 6$ [5] (see also [4], [25] for R_{A_3}) this theorem calculates the polynomial $R_{A_0^i}$, hence determines e.g. the general quadruple point formula. After some preparations, the proof for the case $i=3$ will be given in Section 5.3. The cases $i=4,5,6$ follow similarly, see Section 5.4.

5.1. **Multiple point formulas for germs.** In what follows let us set the cohomology classes in the source and target of the set of j-tuples of points of a map f as $\bar{m}_j(f)$, and $\bar{n}_j(f)$ respectively. That is, $\bar{m}_j(f) = \bar{m}_{A_0^j}(f)$ and $\bar{n}_j(f) = \bar{n}_{A_0^j}(f)$. We also use n_1 for $\bar{n}_1$. Using these notations the defining equations (3) of $R_{A_0^i}$'s can be brought to the following form

$$\bar{m}_2(f) = f^*(\bar{n}_1(f)) + R_{A_0^2}(\ell) \tag{8}$$

$$\bar{m}_3(f) = f^*(\bar{n}_2(f)) + R_{A_0^2}(\ell)f^*(\bar{n}_1(f)) + \frac{1}{2}R_{A_0^3}(\ell) \tag{9}$$

$$\bar{m}_4(f) = f^*(\bar{n}_3(f)) + R_{A_0^2}(\ell)f^*(\bar{n}_2(f)) + \frac{1}{2}R_{A_0^3}(\ell)f^*(\bar{n}_1(f)) + \frac{1}{6}R_{A_0^4}(\ell). \tag{10}$$

We want to apply the method of interpolation from Section 3.1, hence we want to apply equations (8)-(10) for stable germs with relative dimension ℓ, whose codimensions do not exceed the codimension of the relevant $\bar{m}_i$. For stable germs, however, more information is available for some of the ingredients. The following lemma is also seen in [22]. We give a proof here for completeness.

Lemma 5.2. *Let f be a stable germ with relative dimension ℓ; and let G be a symmetry group of f with representations ρ_0 and ρ_1 on the source and target spaces respectively. For a G-representation ρ let $e(\rho)$ denote the G-equivariant Euler class of ρ, that is, the product of the weights of ρ. Then in G-equivariant cohomology we have*

- *f^* is isomorphism;*
- *$f^*(n_1)e(\rho_0) = f^*(e(\rho_1))$;*
- *$f^*(\bar{n}_r) = \frac{1}{r}\bar{m}_r f^*(n_1)$.*

Proof. The map f is equivariantly homotopic to the map of a one point space to a one point space, hence $f^* : H^*BG \to H^*BG$ is the identity map.

Now recall the adjunction formula for the Gysin map $f_!$ (which holds for any proper map, therefore for any stable map germ too):

$$f_!(f^*(x)y) = xf_!(y).$$

Applying f^* to this formula, and writing z for $f^*(x)$, and substituting $y = 1$ we obtain

$$f^*(f_!(z)) = zf^*(n_1), \tag{11}$$

where we also used that $f_!(1)$ is n_1. Since f^* is an isomorphism (hence surjective) this formula holds for any z.

Observe that $f_!(e(\rho_0)) = e(\rho_1)$. Indeed, the Poincaré dual of $e(\rho_0)$ is the homology class of 0 in the source, its homology push-forward is the homology class 0 in the target, whose Poincaré dual is then $e(\rho_1)$. Therefore substituting $z = e(\rho_0)$ in (11) we obtain the second statement of the lemma.

Observe that $f_!(\bar{m}_r) = r\bar{n}_r$. Therefore substituting $z = \bar{m}_r$ into (11) we obtain the third statement. □

Remark 5.3. Since f^* is an isomorphism for germs, we will sometimes suppress it from the notation. Observe that if $e(\rho_0) \neq 0$ then the second statement can be rewritten as $f^*(n_1) = e(\nu(f))$, the equivariant Euler class of the virtual normal bundle. The divisibility of $e(\rho_1)$ with $e(\rho_0)$ is a remarkable property of stable germs. For instance it does not hold for the non-proper blow-up map $(x, y) \to (x, xy)$ with group $U(1) \times U(1)$ acting via $\rho_0 = \alpha \oplus \beta$, $\rho_1 = \alpha \oplus (\alpha \otimes \beta)$.

Using the statements of Lemma 5.2 we can bring formulas (8)-(10) to the forms

$$\bar{m}_2(f) = R_{A_0^2}(\ell) + n_1, \tag{12}$$

$$\bar{m}_3(f) = \frac{1}{2}R_{A_0^3}(\ell) + n_1\big(\dots\big), \tag{13}$$

$$\bar{m}_4(f) = \frac{1}{6}R_{A_0^4}(\ell) + n_1\big(\dots\big), \tag{14}$$

where $n_1(\dots)$ stands for a term divisible by n_1.

We will use these formulas to calculate certain substitutions of residue polynomials. The variables of these polynomials are $c_1, c_2, \ldots$. We will use the following notation for polynomials p with those variables: $p(1 + x_1 + x_2 + \ldots)$ will denote the substitution $c_1 = x_1$, $c_2 = x_2, \ldots$. Furthermore, the series $1 + x_1 + x_2 + \ldots$ will be usually given by (the Taylor series of) a rational function. For example,

$$p(\frac{1+2\alpha}{1+\alpha})$$

means the polynomial p with substitution $c_1 = \alpha$, $c_2 = -\alpha^2$, $c_3 = \alpha^3$, etc.

5.2. **Some stable singularities and their symmetries.** Along the way of proving Theorem 5.1 we will need the following stable singularities.

- A stable A_1 singularity is $f_{A_1} = f_{A_1}(\ell) : \mathbb{C}^{\ell+1}, 0 \to \mathbb{C}^{2\ell+1}, 0$:

$$f_{A_1} : (x, y_1, \ldots, y_\ell) \mapsto (x^2, xy_1, \ldots, xy_\ell, y_1, \ldots, y_\ell).$$

Just like in Section 3.2, we consider its maximal compact symmetry group $G = U(1) \times U(\ell)$ with the representations

$$\rho_1 \oplus (\overline{\rho_1} \otimes \rho_\ell), \qquad \rho_1^2 \oplus \rho_\ell \oplus (\overline{\rho_1} \otimes \rho_\ell)$$

on the source and target spaces. For the G-equivariant cohomology ring we have,

$$H^*BG \le \mathbb{Q}[\alpha, \beta_1, \ldots, \beta_\ell],$$

where α, and β_i's are the Chern roots of the groups $U(1)$, and $U(\ell)$. Using this notation the total Chern class of the virtual normal bundle of f_{A_1} is

$$c(f_{A_1}) = \frac{(1+2\alpha)\prod^\ell (1+\beta_i)}{(1+\alpha)}.$$

Below is the list of the analogous data for singularities A_2, $III_{2,2}$, and A_3.

- A stable A_2 singularity is $f_{A_2} = f_{A_2}(\ell) : \mathbb{C}^{2\ell+2}, 0 \to \mathbb{C}^{3\ell+2}, 0$:

$$f_{A_2} : (x, a, y_1, \ldots, y_\ell, z_1, \ldots, z_\ell) \mapsto (x^3 + xa, x^2 y_1 + xz_1, \ldots, x^2 y_\ell + xz_\ell, a, y_1, \ldots, y_\ell, z_1, \ldots z_\ell).$$

Its maximal compact symmetry group $G = U(1) \times U(\ell)$ acts by the the representations

$$\rho_1 \oplus \rho_1^2 \oplus (\overline{\rho_1}^2 \otimes \rho_\ell) \oplus (\overline{\rho_1} \otimes \rho_\ell) \qquad \rho_1^3 \oplus \rho_\ell \oplus \rho_1^2 \oplus (\overline{\rho_1}^2 \otimes \rho_\ell) \oplus (\overline{\rho_1} \otimes \rho_\ell)$$

on the source and target spaces. For the G-equivariant cohomology ring we have,

$$H^*BG \le \mathbb{Q}[\alpha, \beta_1, \ldots, \beta_\ell],$$

where α, and β_i's are the Chern roots of the groups $U(1)$, and $U(\ell)$. Using this notation the total Chern class of the virtual normal bundle of f_{A_2} is

$$c(f_{A_2}) = \frac{(1+3\alpha)\prod^\ell (1+\beta_i)}{(1+\alpha)}.$$

- A stable $III_{2,2}$ singularity is $f_{III_{2,2}} = f_{III_{2,2}}(\ell) : \mathbb{C}^{2\ell+4}, 0 \to \mathbb{C}^{3\ell+4}, 0$:

$$f_{III_{2,2}} : (x_1, x_2, a, b, c, d, y_1, \ldots, y_{\ell-1}, z_1, \ldots, z_{\ell-1}) \mapsto$$

$$(x_1x_2, x_1^2 + cx_1 + ax_2, x_2^2 + bx_1 + dx_2, y_1x_1 + z_1x_2, \ldots, y_{\ell-1}x_1 + z_{\ell-1}x_2, a, b, c, d, y_1, \ldots, y_{\ell-1}, z_1, \ldots, z_{\ell-1})$$

We consider its symmetry group $G = U(1) \times U(1) \times U(\ell - 1)$ with the representations

$$\rho_1 \oplus \rho_2 \oplus (\rho_1^2 \otimes \overline{\rho_2}) \oplus (\overline{\rho_1} \otimes \rho_2^2) \oplus \rho_1 \oplus \rho_2 \oplus (\rho_\ell \otimes \overline{\rho_1}) \oplus (\rho_\ell \otimes \overline{\rho_2}),$$

$$(\rho_1 \otimes \rho_2) \oplus \rho_1^2 \oplus \rho_2^2 \oplus \rho_\ell \oplus (\rho_1^2 \otimes \overline{\rho_2}) \oplus (\overline{\rho_1} \otimes \rho_2^2) \oplus \rho_1 \oplus \rho_2 \oplus (\rho_\ell \otimes \overline{\rho_1}) \oplus (\rho_\ell \otimes \overline{\rho_2})$$

on the source and target spaces. For the G-equivariant cohomology ring we have,

$$H^*BG \leq \mathbb{Q}[\alpha_1, \alpha_2, \beta_1, \ldots, \beta_{\ell-1}],$$

where α_i, and β_j's are the Chern roots of the respective $U(1)$ groups, and of the group $U(\ell-1)$. Using this notation the total Chern class of the virtual normal bundle of $f_{III_{2,2}}$ is

$$c(f_{III_{2,2}}) = \frac{(1+2\alpha_1)(1+2\alpha_2)(1+(\alpha_1+\alpha_2))\prod^{\ell-1}(1+\beta_i)}{(1+\alpha_1)(1+\alpha_2)}.$$

- A stable A_3 singularity is $f_{A_3} = f_{A_3}(\ell) : \mathbb{C}^{3\ell+3}, 0 \to \mathbb{C}^{4\ell+3}, 0$:

$$f_{A_3} : (x, a, b, w_1, \ldots, w_\ell, y_1, \ldots, y_\ell, z_1, \ldots, z_\ell) \mapsto$$

$$(x^4 + x^2 a + xb, x^3 w_1 + x^2 y_1 + x z_1, \ldots, x^3 w_\ell + x^2 y_\ell + x z_\ell, a, b, w_1, \ldots, w_\ell, y_1, \ldots, y_\ell, z_1, \ldots z_l).$$

We consider its symmetry group $G_f = U(1) \times U(\ell)$ with the representations

$$\rho_1 \oplus \rho_1^2 \oplus \rho_1^3 \oplus (\overline{\rho_1}^3 \otimes \rho_\ell) \oplus (\overline{\rho_1^2} \otimes \rho_\ell) \oplus (\overline{\rho_1} \otimes \rho_\ell)$$

$$\rho_1^4 \oplus \rho_\ell \oplus \rho_1^2 \oplus \rho_1^3 \oplus (\overline{\rho_1}^3 \otimes \rho_\ell) \oplus (\overline{\rho_1^2} \otimes \rho_\ell) \oplus (\overline{\rho_1} \otimes \rho_\ell)$$

on the source and target spaces. The G-equivariant cohomology ring

$$H^*BG \leq \mathbb{Q}[\alpha, \beta_1, \ldots, \beta_\ell],$$

where α, and β_i's are the Chern roots of the groups $U(1)$, and $U(\ell)$. Using this notation the total Chern class of the virtual normal bundle of f_{A_3} is

$$c(f_{A_3}) = \frac{(1+4\alpha)\prod^{\ell}(1+\beta_i)}{(1+\alpha)}.$$

5.3. **Proof of Theorem 5.1.** Now we prove Theorem 5.1 for $i = 3$, that is

$$R_{A_0^4}(\ell) = -6R_{A_3}(\ell - 1). \tag{15}$$

Proof. Consider the stable singularity of type A_1 with relative dimension ℓ from Section 5.2. This map f_{A_1} has no quadruple point, hence $\bar{m}_4 = 0$ for it. This can be checked directly, or using the fact from singularity theory that the highest multiple points of a stable singularity with local algebra Q are the δ-tuple points, where δ is the dimension of the local algebra Q.

Lemma 5.2 above shows that for f_{A_1} we have

$$n_1(f_{A_1}) = \frac{2\alpha \prod^{\ell}\beta_i \prod^{\ell}(\beta_i - \alpha)}{\alpha \prod^{\ell}(\beta_i - \alpha)} = 2\prod^{\ell}\beta_i.$$

Thus, for f_{A_1} equation (14) becomes

$$0 = \frac{1}{6}R_{A_0^4}(\ell)\big(\frac{(1+2\alpha)\prod^{\ell}(1+\beta_i)}{(1+\alpha)}\big) + \big(2\prod^{\ell}\beta_i\big)\big(\ldots\big).$$

Plugging in $\beta_\ell = 0$ we obtain

$$0 = R_{A_0^4}(\ell)\big(\frac{(1+2\alpha)\prod^{\ell-1}(1+\beta_i)}{(1+\alpha)}\big). \tag{16}$$

We repeat the above arguments for the stable singularity of type A_2 with relative dimension ℓ, and we obtain

$$0 = R_{A_0^4}(\ell)\big(\frac{(1+3\alpha)\prod^{\ell-1}(1+\beta_i)}{(1+\alpha)}\big). \tag{17}$$

The argument for the $III_{2,2}$ singularity is similar. We have

$$n_1(f_{III_{2,2}}) = \frac{4\alpha_1^2\alpha_2^2(\alpha_1+\alpha_2)(2\alpha_1-\alpha_2)(2\alpha_2-\alpha_1)\prod^{\ell-1}(\beta_i(\beta_i-\alpha_1)(\beta_i-\alpha_2))}{\alpha_1^2\alpha_2^2(2\alpha_1-\alpha_2)(2\alpha_2-\alpha_1)\prod^{\ell-1}((\beta_i-\alpha_1)(\beta_i-\alpha_2))} = 4(\alpha_1+\alpha_2)\prod^{\ell-1}\beta_i.$$

Thus for $f_{III_{2,2}}$ equation (14) becomes

$$0 = \frac{1}{6}R_{A_0^4(\ell)}\big(\frac{(1+2\alpha_1)(1+2\alpha_2)(1+(\alpha_1+\alpha_2))\prod^{\ell-1}(1+\beta_i)}{(1+\alpha_1)(1+\alpha_2)}\big) + 4(\alpha_1+\alpha_2)\prod^{\ell-1}\beta_i\big(\dots\big).$$

Substituting $\beta_{\ell-1}=0$ we obtain

$$0 = R_{A_0^4(\ell)}\big(\frac{(1+2\alpha_1)(1+2\alpha_2)(1+(\alpha_1+\alpha_2))\prod^{\ell-2}(1+\beta_i)}{(1+\alpha_1)(1+\alpha_2)}\big). \tag{18}$$

Now consider the stable singularity of type A_3 with relative dimension ℓ from Section 5.2. The closure of the quadruple point set of f_{A_3} in the source space is $\{w_i=0, y_i=0, z_i=0\}$. Hence for f_{A_3} we have $\bar{m}_4 =$ Euler class of the normal bundle to $\{w_i=0, y_i=0, z_i=0\}$. That is

$$\bar{m}_4 = \prod^{\ell}(\beta_i-\alpha)(\beta_i-2\alpha)(\beta_i-3\alpha).$$

Lemma 5.2 above shows that for f_{A_3} we have

$$n_1(f_{A_3}) = 4\prod^{\ell}\beta_i.$$

Thus, for f_{A_3} equation (14) becomes

$$\prod^{\ell}(\beta_i-\alpha)(\beta_i-2\alpha)(\beta_i-3\alpha) = \frac{1}{6}R_{A_0^4}(\ell)\big(\frac{(1+4\alpha)\prod^{\ell}(1+\beta_i)}{(1+\alpha)}\big) + 4\prod^{\ell}\beta_i\big(\dots\big).$$

Plugging in $\beta_\ell = 0$ we obtain

$$-6\alpha^3\prod^{\ell-1}(\beta_i-\alpha)(\beta_i-2\alpha)(\beta_i-3\alpha) = \frac{1}{6}R_{A_0^4(\ell)}\big(\frac{(1+4\alpha)\prod^{\ell-1}(1+\beta_i)}{(1+\alpha)}\big). \tag{19}$$

Observe that formulas (16), (17) (18) and (19) mean that the polynomial $-\frac{1}{6}R_{A_0^4}(\ell)$ satisfies the following properties: (i) it vanishes when applied to $f_{A_1}(\ell-1)$, $f_{A_2}(\ell-1)$, $f_{III_{2,2}}(\ell-1)$; (ii) it gives the Euler class of the source space when applied to $f_{A_3}(\ell-1)$. These are exactly the properties of the polynomial $R_{A_3}(\ell-1)$ applied to these four singularities. According to Theorem 3.2, these properties determine $R_{A_3}(\ell-1)$, hence we have proven that $R_{A_0^4}(\ell) = -6R_{A_3}(\ell-1)$. $\square$

In summary we obtained the *general quadruple point formula*:

$$m_4 = f^*(n_3) - 3c_\ell f^*(n_2) + 6\Big(c_\ell^2 + \sum_{i=0}^{\ell-1}2^i c_{\ell-1-i}c_{\ell+1+i}\Big)f^*(n_1) + p(c_i) \tag{20}$$

where

$$p(c_i)=R_{A_0^4}=-6\left(\sum_{i=0}^{\infty}2^i c_{\ell-i}c_\ell c_{\ell+i}+\frac{1}{3}\sum_{i=1}^{\infty}\sum_{j=1}^{\infty}2^i3^jc_{\ell-i}c_{\ell-j}c_{\ell+i+j}+\frac{1}{2}\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}a_{i,j}c_{\ell-i-j}c_{\ell+i}c_{\ell+j}\right)$$

where $c_0=1, c_{<0}=0$ and with the $a_{i,j}$'s defined as for the Thom polynomial of A_3, that is:

$$\sum_{i,j}a_{i,j}u^iv^j=\frac{u\frac{1-u}{1-3u}+v\frac{1-v}{1-3v}}{1-u-v}.$$

Remark 5.4. Another way of viewing the $a_{i,j}$'s is as the entries of the following, modified Pascal's triangle

$$\begin{array}{ccccccccc} &&&& a_{0,0} &&&& \\ &&& a_{1,0} && a_{0,1} &&& \\ && a_{2,0} && a_{1,1} && a_{0,2} && \\ & a_{3,0} && a_{2,1} && a_{1,2} && a_{0,3} & \\ a_{4,0} && a_{3,1} && a_{2,2} && a_{1,3} && a_{0,4} \end{array} = \begin{array}{ccccccccc} &&&& 0 &&&& \\ &&& 1 && 1 &&& \\ && 3 && 2 && 3 && \\ & 9 && 5 && 5 && 9 & \\ 27 && 14 && 10 && 14 && 27 \end{array}$$

where the rule for the i,jth entry remains the same, but we have placed powers of 3 on the edges instead of 1's.

5.4. **Higher multiple point formulas.** The proof of Theorem 5.1 for $i=4,5,6$ goes along the same line as for $i=3$. One considers the finitely many monosingularities whose codimension is less than $(i+1)\ell$, as well as the monosingularity $A_i(\ell)$. Applying the defining relation of $R_{A_0^{i+1}}$ for these monosingularities (in equivariant cohomology) results in certain formulas for different specializations of $R_{A_0^{i+1}}$. Plugging in 0 for the "last Chern root", just like in (16), one obtains some shorter, simpler formulas, which turn out to mean that the residue polynomial $(-1)^i/i!\cdot R_{A_0^{i+1}}(\ell)$ satisfies the exact same substitutions as $R_{A_i}(\ell-1)$. Using the statement that these substitutions determine $R_{A_i}(\ell-1)$ (Theorem 3.2) we conclude that $R_{A_0^{i+1}}(\ell)=(-1)^i i! R_{A_i}(\ell-1)$.

One naturally conjectures that $R_{A_0^{i+1}}(\ell)=(-1)^i i! R_{A_i}(\ell-1)$ holds for all i. We found reasons supporting this conjecture, but no proof. The method this article uses certainly does not work for $i>6$. The reason is a 1-dimensional family of singularities that together form a codimension $6\ell+9$ variety in $\mathcal{E}^0(n,n+\ell)$ (for large ℓ) [27]. Hence, beyond codimension $6\ell+8$ we can not apply Theorem 3.2.

6. 4-secants to Smooth Projective Varieties

As an application of the quadruple point formula we find the number of 4-secant planes to smooth projective varieties. The method can be tailored to find the number of 4-secant (or 5-, 6-, or 7-secant) linear spaces of other dimensions.

Let $i: V^a\subset\mathbb{P}^{4a+2}$ be a smooth projective variety, and let $\mathbf{G}=\mathrm{Gr}_2\,\mathbb{P}^{4a+2}=\mathrm{Gr}_3\,\mathbb{C}^{4a+3}$ denote the Grassmannian of projective 2-planes in $\mathbb{P}^{4a+2}$. Consider the following incidence varieties:

$$\begin{aligned}\mathbf{B} &:= \{(x,P)\in V\times\mathbf{G}\mid x\in P\},\\ \mathbf{F} &:= \{(x,P)\in\mathbb{P}^{4a+2}\times\mathbf{G}\mid x\in P\}.\end{aligned}$$

The two projections of $\mathbf{F}$ to $\mathbb{P}^{4a+2}$ and $\mathbf{G}$ will be denoted by q and π. Both are fibrations with fibers $\mathrm{Gr}_2\,\mathbb{C}^{4a+2}$ and $\mathbb{P}^2$, respectively. The restriction of q to the variety V is the fibration $p:\mathbf{B}\to V$. Hence we obtain the following diagram.

$$\begin{array}{ccccc} \mathbf{B}^{9a} & \xrightarrow{j} & \mathbf{F}^{12a+2} & \xrightarrow{\pi} & \mathbf{G}^{12a} \\ \Big\downarrow p & & \Big\downarrow q & & \\ V^a & \xrightarrow{i} & \mathbb{P}^{4a+2}, & & \end{array} \tag{21}$$

where upper indexes mean dimensions. Observe that the quadruple points of the map $f = \pi \circ j$ correspond bijectively to planes intersecting V exactly four times, ie. 4-secant planes.

We will make the assumption that the map f is admissible (cf. Remark 6.3). Hence, the number N_a of 4-secant planes to V is calculated as

$$N_a = \frac{1}{4!}\int_{\mathbf{G}} n_{A_0^4(f)} \tag{22}$$

$$= \frac{1}{4!}\int_{\mathbf{G}} f_!(R_{A_0})^4 + 6f_!(R_{A_0^2})f_!(R_{A_0})^2 + 3f_!(R_{A_0^2})^2 + 4f_!(R_{A_0^3})f_!(R_{A_0}) + f_!(R_{A_0^4}), \tag{23}$$

where the polynomials $R_{A_0^i}$ are evaluated at the Chern classes of the virtual normal bundle ν_f of f. In the rest of this section we show how this integral can be calculated.

First observe that $\nu_f = \nu_j \oplus j^*\nu_\pi = p^*(\nu_i) \ominus j^*(\kappa)$, where κ is the fiberwise tangent bundle to the fibration π. Let the Chern classes of κ be k_1, k_2, and let the Chern classes of ν_i be $n_1, \ldots, n_a$.

Since the the polynomials $R_{A_0^i}$ are explicitly known (see Section 4), the integrand in (23) is an explicit polynomial, whose terms are of the form $f_!(j^*(k)p^*(n))$, where k is a monomial in k_1, k_2, and n is a monomial in $n_1, \ldots, n_a$. This term is further equal to

$$\pi_! j_!(j^*(k)p^*(n)) = \pi_!(kj_!p^*(n)) = \pi_!(kq^*(i_!(n))).$$

The cohomology classes $i_!(n)$ are the geometric invariants of the variety $V^a \subset \mathbb{P}^{4a+2}$—we want to calculate the number of 4-secant planes in terms of these invariants. These classes can be encoded by integers, as follows.

Definition 6.1. *Let h be the class represented by a hyperplane in $H^*(\mathbb{P}^{4a+2})$, hence $H^*(\mathbb{P}^{4a+2}) = \mathbb{Q}[h]/(h^{4a+3})$. For a multiindex $u = (u_1, u_2, \ldots, u_a)$ let χ_u be the coefficient of the appropriate power of h in $i_!(n_1^{u_1}n_2^{u_2}\cdots n_a^{u_a})$. (For example $\chi_{(0,\ldots,0)}$ is the degree of the embedding $V \subset \mathbb{P}^{4a+2}$.)*

Using this notation, we obtain that our integrand can be written as a linear combination of terms of the form $\pi_!(k \cdot q^*(h^w))$ ($w \in \mathbb{N}$), with coefficients depending on the invariants χ_u.

Let S and Q be the universal sub and quotient bundles over $\mathbf{G}$. The space $\mathbf{F}$ is the projectivization of the bundle S. Corresponding to this fact, we have the tautological exact sequence of bundles $0 \to l \to \pi^*S \to \pi^*S/l \to 0$ over F. Moreover, κ being the fiberwise tangent bundle, we have $\kappa = l^* \otimes \pi^*S/l$. Using the fact that $q^*(h)$ is the first Chern class of l, we obtain that the integrand can further be written as linear combination of terms of the form

$$\pi_!(c_1(l)^w c_I(\pi^*(S))).$$

Here c_I is any Chern monomial, and w is a non-negative integer. This term is further equal to

$$c_I(S)\pi_!(c_1(l)^w).$$

The cohomology ring of $\mathbf{G}$, together with the $\pi_!$-image of powers of $c_1(l)$ are well known, see for example [16]:

$$H^*(\mathbf{G}) = Q[c_i(S), c_i(Q)]/(c(S)c(Q) = 1),$$

$$\pi_!(c_1(l)^w) = c_{w-2}(Q).$$

Hence our integrand is an explicit class in $H^*(\mathbf{G})$. Integration can be utilized in any computer algebra package. The results we obtain this way are as follows.

Theorem 6.2. *Let $V^a \subset \mathbb{P}^{4a+2}$ be a smooth variety such that the associated map $f = \pi \circ j : \mathbf{B} \to \mathbf{G}$ defined in (21) is admissible. Let χ_u be the invariants of the embedding. Then for the number N_a of 4-secant planes to V^a we have*

$$4!N_1 = {\chi_0}^4 + 24\chi_1\chi_0 - 6\chi_1{\chi_0}^2 - 208{\chi_0}^2 + 24{\chi_0}^3 + 3{\chi_1}^2 + 1008\chi_0 - 174\chi_1,$$

$$\begin{aligned}4!N_2 = {} & -36\chi_{1,0}\chi_{0,1} + 64\chi_{2,0}\chi_{0,0} - 3156\chi_{1,0}\chi_{0,0} + {\chi_{0,0}}^4 + 36\chi_{1,0}{\chi_{0,0}}^2 \\ & -6\chi_{0,1}{\chi_{0,0}}^2 - 126{\chi_{0,0}}^3 + 12075{\chi_{0,0}}^2 + 286\chi_{0,0}\chi_{0,1} - 1356\chi_{2,0} \\ & -1944\chi_{0,1} - 200838\chi_{0,0} + 3{\chi_{0,1}}^2 + 108{\chi_{1,0}}^2 + 42174\chi_{1,0},\end{aligned}$$

$$\begin{aligned}4!N_3 = {} & -1728\chi_{1,0,0}\chi_{0,1,0} + 91200\chi_{0,0,0}\chi_{1,0,0} - 6\chi_{0,0,1}{\chi_{0,0,0}}^2 + 384\chi_{1,1,0}\chi_{0,0,0} \\ & -48\chi_{0,1,0}\chi_{0,0,1} - 144\chi_{0,0,0}\chi_{0,0,1} - 4352\chi_{2,0,0}\chi_{0,0,0} - 26004\chi_{1,1,0} + \\ & 3{\chi_{0,0,1}}^2 - 9523080\chi_{1,0,0} + {\chi_{0,0,0}}^4 + 42058080\chi_{0,0,0} + 614880\chi_{0,1,0} \\ & -23934\chi_{0,0,1} - 3156\chi_{3,0,0} - 448320{\chi_{0,0,0}}^2 + 437400\chi_{2,0,0} + 3888{\chi_{1,0,0}}^2 \\ & +192{\chi_{0,1,0}}^2 + 720{\chi_{0,0,0}}^3 - 5120\chi_{0,0,0}\chi_{0,1,0} + 216\chi_{1,0,0}\chi_{0,0,1} \\ & -216\chi_{1,0,0}{\chi_{0,0,0}}^2 + 48\chi_{0,1,0}{\chi_{0,0,0}}^2,\end{aligned}$$

$$\begin{aligned}4!N_4 = {} & 1280\chi_{1,0,1,0}\chi_{0,0,0,0} - 1320\chi_{0,0,0,1}\chi_{1,0,0,0} + 853550\chi_{0,1,0,0}\chi_{0,0,0,0} \\ & +1320\chi_{1,0,0,0}{\chi_{0,0,0,0}}^2 - 33024\chi_{1,1,0,0}\chi_{0,0,0,0} + 60\chi_{0,0,1,0}{\chi_{0,0,0,0}}^2 \\ & -72600\chi_{0,1,0,0}\chi_{1,0,0,0} - 330\chi_{0,1,0,0}{\chi_{0,0,0,0}}^2 - 3300\chi_{0,0,1,0}\chi_{0,1,0,0} \\ & -60\chi_{0,0,1,0}\chi_{0,0,0,1} + 6466\chi_{0,0,0,1}\chi_{0,0,0,0} - 4290{\chi_{0,0,0,0}}^3 \\ & -6721080\chi_{1,0,0,0}\chi_{0,0,0,0} - 92436\chi_{0,0,1,0}\chi_{0,0,0,0} + 247040\chi_{2,0,0,0}\chi_{0,0,0,0} \\ & +512\chi_{0,2,0,0}\chi_{0,0,0,0} + 330\chi_{0,0,0,1}\chi_{0,1,0,0} - 309768\chi_{0,0,0,1} \\ & +2126696220\chi_{1,0,0,0} + 13200\chi_{0,0,1,0}\chi_{1,0,0,0} + 300{\chi_{0,0,1,0}}^2 \\ & +1272924\chi_{3,0,0,0} + 9382770\chi_{0,0,1,0} - 9023984640\chi_{0,0,0,0} + 10379016\chi_{1,1,0,0} \\ & +3{\chi_{0,0,0,1}}^2 - 104832\chi_{0,2,0,0} + 145200{\chi_{1,0,0,0}}^2 - 158489298\chi_{0,1,0,0} \\ & -292860\chi_{1,0,1,0} + 9075{\chi_{0,1,0,0}}^2 + {\chi_{0,0,0,0}}^4 - 81576\chi_{2,1,0,0} \\ & -113973552\chi_{2,0,0,0} + 24962795{\chi_{0,0,0,0}}^2 - 6\chi_{0,0,0,1}{\chi_{0,0,0,0}}^2.\end{aligned}$$

We note that the expression for N_2 has appeared in [37] and [26], in the language of Hilbert schemes (and in the variables $d = \chi_{0,0}$, $\pi = \chi_{1,0} - 11\chi_{0,0}$, $\kappa = \chi_{2,0} - 22\chi_{1,0} + 121\chi_{0,0}$, $e = -\chi_{0,1} + \chi_{2,0} - 11\chi_{1,0} + 55\chi_{0,0}$), but we believe that N_3 and beyond are new results. Expressions for $N_{>4}$, as well as formulas counting 4-secant linear spaces of higher dimensions can be obtained similarly.

Remark 6.3. Theorem 6.2 contains the unpleasant condition that the associated map is admissible. Looking through the literature on enumerative geometry using topological methods we find that authors explicitly or implicitly suppose similar admissibility properties. Namely, the following seems to be a general belief: when starting with a geometric situation one associates a map between parameter spaces, and the map is not a Legendre or Lagrange map (e.g. its relative dimension is >1), then the map is admissible, provided some genericity condition holds on the original map. In simple situations, e.g. linear projections of complex projective varieties in nice dimensions, there is a Bertini-type theorem due to Mather (see [29, 28, 1]), which may be applied to prove the admissibility property of the associated map $f=\pi\circ j$ of diagram (21).

7. Another Multisingularity Formula

The interpolation method described in Section 3.1 can be applied to find finite initial sums of the series describing the general multisingularity polynomials. If the multisingularity is complicated enough, recognizing and proving the pattern in such final sums quickly become intractable. An exception is given by the theorem below. We will use the following versions of Schur polynomials

$$s(i,j,k)=\det\begin{pmatrix} c_i & c_{i+1} & c_{i+2}\\ c_{j-1} & c_j & c_{j+1}\\ c_{k-2} & c_{k-1} & c_k\end{pmatrix},\qquad s(i,j)=\det\begin{pmatrix} c_i & c_{i+1}\\ c_{j-1} & c_j\end{pmatrix}. \tag{24}$$

Theorem 7.1. *The general $III_{2,2}A_0$-multisingularity residue polynomial for maps of relative dimension ℓ is*

$$R_{III_{2,2}A_0}=-\sum_{i=1}^{\infty}2^{i+1}s(\ell+1+i,\ell+2,\ell+1-i). \tag{25}$$

Proof. Let us denote the right hand side of equation (25) by $\mathcal{R}$. We will show that $\mathcal{R}$ satisfies the defining relation of the residue polynomial $R_{III_{2,2}A_0}$, that is, we will show

$$m_{III_{2,2}A_0}=\mathcal{R}+R_{III_{2,2}}n_{A_0} \tag{26}$$

for all admissible maps. The Giambelli-Thom-Porteous formula states that $R_{III_{2,2}}=s(\ell+2,\ell+2)$. Theorem 3.2 asserts that if (26) holds for stable representatives of A_0, A_1, A_2, A_3, $I_{2,2}$, and $III_{2,2}$ singularities (in equivariant cohomology with respect to the maximal compact symmetry group of the particular singularity), then (26) holds for admissible maps. Below we prove these statements.

7.1. Restriction to A_r singularities.

Stable representatives of ℓ relative dimensional A_r singularities are universal unfoldings of germs $\mathbb{C}\to\mathbb{C}^{\ell+1}$

$$(x)\mapsto(x^{r+1},0,\dots,0).$$

Their maximal compact symmetry group is $U(1)\times U(\ell)$. The formal difference of the representation on the target and the source is

$$\rho_1^{r+1}\oplus\rho_\ell-\rho_1,$$

where ρ_1 and ρ_ℓ are the standard representations of the $U(1)$ and $U(\ell)$ factors. Therefore the Chern classes c_i of the stable representative of A_r are obtained by

$$1+c_1t+c_2t^2+\ldots=\frac{1-(r+1)at}{1-at}\sum_{i=0}^{\ell}d_it^i,\qquad (d_0=1) \tag{27}$$

where $-a$ is the first Chern class of $U(1)$, and d_i are the Chern classes of $U(\ell)$. Observe that relation (27) implies $c_{j+1} = ac_j$ for $j \geq \ell + 2$. Therefore the first two rows of each term of $\mathcal{R}$ are linearly dependent, making each determinant 0. Hence $\mathcal{R} = 0$ applied to any A_r singularity.

Since $III_{2,2}$ is a Σ^2 singularity, and all A_r's are Σ^1 singularities, near an A_r singularity there are no $III_{2,2}$ or $III_{2,2}A_0$ (multi)singularities. This implies that $R_{III_{2,2}} = m_{III_{2,2}}$ and $m_{III_{2,2}A_0}$ applied to stable representatives of all A_r singularities are both 0. Therefore we proved that (26) holds for all A_r singularities.

7.2. **Restriction to $I_{2,2}$ singularities.** Stable singularities of type $I_{2,2}$ of relative dimension ℓ are universal unfoldings of the germ $\mathbb{C}^2 \to \mathbb{C}^{\ell+2}$

$$(x, y) \mapsto (x^2, y^2, 0, \dots, 0).$$

The maximal compact symmetry group of this germ is $U(1)^2 \times U(\ell)$, and the formal difference of the representations of this group on the target and on the source is:

$$\rho_1^2 \oplus \rho_1'^2 \oplus \rho_\ell - (\rho_1 \oplus \rho_1').$$

Here ρ_1 and ρ_1' are the standard representations of the two $U(1)$ factors, and ρ_ℓ is the standard representation of $U(\ell)$. Therefore the Chern classes c_i of the stable representative of $I_{2,2}$ are obtained by

$$1 + c_1 t + c_2 t^2 + \ldots = \frac{(1-2at)(1-2bt)}{(1-at)(1-bt)} \sum_{i=0}^{\ell} d_i t^i, \tag{28}$$

where $-a$ and $-b$ are the first Chern classes of the two $U(1)$ factors, and d_i are the Chern classes of $U(\ell)$. We need the following lemma.

Lemma 7.2. *Let e_i and h_i denote the elementary, and complete symmetric polynomials of the variables a and b (e.g. $e_2 = ab$, $h_2 = a^2 + ab + b^2$). Suppose the variables c_i are expressed in terms of a, b, and $d_1, \dots, d_\ell$ as defined in (28). We use the convention that $d_0 = 1$, $d_{<0} = 0$ and $d_{>\ell} = 0$. Then*

$$s(\alpha, \beta, \gamma) = e_2^{\beta-\ell-2} \cdot h_{\alpha-\beta} \cdot s(\ell+2, \ell+2) \cdot (d_\gamma - 2e_1 d_{\gamma-1} + 4e_2 d_{\gamma-2}),$$

for $\alpha \geq \beta \geq \gamma$, $\beta \geq \ell + 2$, and $\gamma \leq \ell$.

Proof. The Factorization Formula for Schur polynomials (e.g. [17]) claims that substituting

$$1 + c_1 t + c_2 t^2 + \ldots = \frac{\sum_{i=0}^{\ell+2} D_i t^i}{(1-at)(1-bt)}$$

into $s(\alpha, \beta, \gamma)$ yields $e_2^{\beta-\ell-2} h_{\alpha-\beta} s(\ell+2, \ell+2) D_\gamma$. Carrying out the further substitution $\sum_{i=0}^{\ell+2} D_i t^i = (\sum_{i=0}^{\ell} d_i t^i)(1-2at)(1-2bt)$ gives the statement of the lemma. ☐

A special case of this lemma claims that for $j \geq 1$ we have

$$s(\ell+1+j, \ell+2, \ell+1-j) = h_{j-1} \cdot s(\ell+2, \ell+2) \cdot (d_{\ell+1-j} - 2e_1 d_{\ell-j} + 4e_2 d_{\ell-1-j}).$$

Plugging this into the formula for $\mathcal{R}$ we obtain a linear function of the d_i variables. The coefficient of $d_{\ell-k}$ for $k > 0$ is

$$-2^k \cdot 4e_2 h_{k-2} - 2^{k+1}(-2e_1) h_{k-1} - 2^{k+2} h_k.$$

Dividing this expression by -2^{k+2} we obtain $e_2h_{k-2} - e_1h_{k-1} + h_k$, which is the k'th coefficient of the power series

$$(1 - e_1t + e_2t)(1 + h_1t + h_2t^2 + \ldots) = \frac{(1-at)(1-bt)}{(1-at)(1-bt)} = 1,$$

hence it is 0. We obtain that substituting (28) into the expression $\mathcal{R}$ is $-4d_\ell \cdot s(\ell+2, \ell+2)$. Lemma 5.2 implies that $n_{A_0} = (-2a)(-2b)d_\ell/((-a)(-b)) = 4d_\ell$ for the germ $I_{2,2}$. Thus we proved that formula (26) holds for stable representatives of $I_{2,2}$ singularities.

7.3. **Restriction to $III_{2,2}$ singularities.** Stable singularities of type $III_{2,2}$ of relative dimension ℓ are universal unfoldings of the germ $\mathbb{C}^2 \to \mathbb{C}^{\ell+2}$

$$(x, y) \mapsto (x^2, y^2, xy, 0, \ldots, 0).$$

The maximal torus of the maximal compact symmetry group of this germ is $U(1)^2 \times U(\ell-1)$, and the formal difference of the representations of this group on the target and on the source is:

$$\rho_1^2 \oplus \rho_1'^2 \oplus (\rho_1 \otimes \rho_1') \oplus \rho_{\ell-1} - (\rho_1 \oplus \rho_1').$$

Here ρ_1 and ρ_1' are the standard representations of the two $U(1)$ factors, and $\rho_{\ell-1}$ is the standard representation of $U(\ell-1)$. Therefore, the Chern classes c_i of the stable representative of $I_{2,2}$ are obtained by

$$1 + c_1t + c_2t^2 + \ldots = \frac{(1-2at)(1-2bt)(1-(a+b)t)}{(1-at)(1-bt)} \sum_{i=0}^{\ell-1} d_it^i, \tag{29}$$

where $-a$ and $-b$ are the first Chern classes of the two $U(1)$ factors, and d_i are the Chern classes of $U(\ell)$.

This shows that substituting (29) into $\mathcal{R}$ can be obtained by first substituting (28) into $\mathcal{R}$, then plugging in $d_\ell = -(a+b)$. The same holds for the other terms of (26) as well, hence the satisfaction of formula (26) for substitution (29) follows from the fact that it is satisfied for the substitution (28).

The proof of Theroem 7.1 is complete. $\square$

Remark 7.3. One can consider applications of the $III_{2,2}A_0$-formula in enumerative geometry along the lines of Section 6. The outcome of such a calculation is then the number (or cohomology class) of k-planes in $\mathbb{P}^N$ that have two common points with a fixed smooth projective variety $V \subset \mathbb{P}^N$; one common point is a transversal intersection, and the other is a singular one, with singularity $III_{2,2}$.

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