Birman's Conjecture is True for $I_2(p)$

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Abstract

In 1993, Birman conjectured that the desingularization map from the singular braid monoid to the integral group ring of the braid group determined by $\sigma_i^{\pm 1} \mapsto \sigma_i^{\pm 1}$ and $\tau_i \mapsto \sigma_i - \sigma_i^{-1}$ is injective. The conjecture, which has recently been proven true by Paris (2003), may be generalised to all Artin groups. In this article we prove that the conjecture holds for one of the infinite families of Artin groups of spherical type, namely $I_2(p)$.

1 Introduction

A Coxeter graph Γ is a complete graph on a finite vertex set S, whose edges are labelled from the set $\{2, 3, \ldots, \infty\}$. For $s, t \in S$, let m_{st} denote the label on the edge $\{s, t\}$, and for convenience let $m_{ss} = \infty$ for all $s \in S$. The *Coxeter group of type* Γ is

$$W_\Gamma = \big\langle S \,\big|\, s^2 = 1 \ \ (\forall s \in S)\,,\ \langle st\rangle^{m_{st}} = \langle ts\rangle^{m_{st}} \text{ if } m_{st} \neq \infty \ \ (\forall s, t \in S)\big\rangle.$$

Here $\langle gh\rangle^m$ denotes the alternating product $ghg\cdots$ with m terms if $m \in \mathbb{N}$, or $\big(\langle gh\rangle^{-m}\big)^{-1}$ if $-m \in \mathbb{N}$. Let $\Sigma = \{\sigma_s \,|\, s \in S\}$ be a set in one-one correspondence with S. The *Artin group of type* Γ is

$$\mathcal{B}_\Gamma = \big\langle \Sigma \,\big|\, \langle \sigma_s\sigma_t\rangle^{m_{st}} = \langle \sigma_t\sigma_s\rangle^{m_{st}} \text{ if } m_{st} \neq \infty \ \ (\forall s, t \in S)\big\rangle.$$

There is a natural surjective homomorphism $\pi : \mathcal{B}_\Gamma \to W_\Gamma : \sigma_s \mapsto s$, the kernel of which, $\mathcal{P}_\Gamma$, is known as the *pure Artin group of type* Γ. Put $\Sigma^{-1} = \{\sigma_s^{-1} \,|\, s \in S\}$ and $T = \{\tau_s \,|\, s \in S\}$. The *singular Artin monoid of type* Γ is the monoid $\mathcal{SB}_\Gamma$ generated by $\Sigma \cup \Sigma^{-1} \cup T$ subject to defining relations, for all $s, t \in S$,

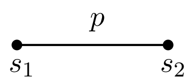
$$\begin{aligned}
\sigma_s^{\pm 1}\sigma_s^{\mp 1} &= 1 & & \\
\sigma_s\tau_s &= \tau_s\sigma_s & & \\
\tau_s\tau_t &= \tau_t\tau_s & &\text{if } m_{st} = 2\\
\langle \sigma_s\sigma_t\rangle^{m_{st}} &= \langle \sigma_t\sigma_s\rangle^{m_{st}} & &\text{if } m_{st} \neq \infty\\
\tau_s\langle \sigma_t\sigma_s\rangle^{m_{st}-1} &= \langle \sigma_t\sigma_s\rangle^{m_{st}-1}\tau_u & &\text{if } m_{st} \neq \infty,
\end{aligned}$$

where $u = s$ if m_{st} is even, or $u = t$ if m_{st} is odd. The τ_s are known as the singular generators. See [5] for more details. The map π extends to a monoid homomorphism $\pi : \mathcal{SB}_\Gamma \to W_\Gamma$ by further defining $\pi(\tau_s) = s$. Analogously, the *pure singular Artin monoid of type* Γ is $\mathcal{SP}_\Gamma = \pi^{-1}(1) = \big\{\beta \in \mathcal{SB}_\Gamma \,\big|\, \pi(\beta) = 1\big\}$.

An important class of Artin groups are those of *spherical* type (ie. the associated Coxeter group is finite). For a classification of finite Coxeter groups see for example [9].

When $\Gamma = A_{n-1}$, $W_\Gamma = \mathfrak{S}_n$ is the symmetric group on n letters [11], $\mathcal{B}_\Gamma = \mathcal{B}_n$ is the n-string braid group [2], and $\mathcal{SB}_\Gamma = \mathcal{SB}_n$ is the n-string singular braid monoid [3, 4].

Denote by $\mathbb{Z}\mathcal{B}_\Gamma$ the integral group ring of the Artin group $\mathcal{B}_\Gamma$. For $s \in S$, define $\eta(\sigma_s^{\pm 1}) = \sigma_s^{\pm 1}$ and $\eta(\tau_s) = \sigma_s - \sigma_s^{-1}$. Then η extends to a well defined homomorphism $\eta : \mathcal{SB}_\Gamma \to \mathbb{Z}\mathcal{B}_\Gamma$. Birman [4] conjectured that η is injective for $\Gamma = A_n$. This is simple when $n = 1$. Járai [10] demonstrated that the conjecture holds when $n = 2$. Recently, Paris [12] proved the conjecture for all n, and Godelle and Paris [8] have proved the conjecture in the case that Γ is *right angled* (ie. $m_{st} \in \{2, \infty\}$ for all s, t). With this in mind, it is natural to ask for which other types the conjecture holds. The purpose of this article is to show how to extend the methods of Járai, and demonstrate the injectivity of η when $\Gamma = I_2(p)$ for any $3 \leq p \in \mathbb{N}$. Along the way we prove some structural results about $\mathcal{SB}_\Gamma$ for arbitrary Γ.

$I_2(p) \;=$ 

2 Preliminary Results

We begin by defining a number of natural homomorphisms which will be used throughout. Define

$$N : \mathcal{SB}_\Gamma \to (\mathbb{N}, +) : \sigma_s^{\pm 1} \mapsto 0\,,\ \tau_s \mapsto 1.$$

So if $\beta \in \mathcal{SB}_\Gamma$, then $N(\beta)$ is the number of τ's in (any word representing) β. Define

$$\phi : \mathcal{SB}_\Gamma \to \mathcal{B}_\Gamma : \sigma_s^{\pm 1} \mapsto \sigma_s^{\pm 1}\,,\ \tau_s \mapsto \sigma_s.$$

We now introduce some notation, and recall some well known properties of Coxeter groups which may be found in any standard text (eg. [9]). Let S^* denote the set of all words over S. If $w, w' \in S^*$, we write $w \equiv w'$ if w and w' are identical words, and $w = w'$ if w and w' represent the same element of W_Γ. For $w \equiv s_1 \cdots s_k \in S^*$, we define $\ell(w) = k$, and define

$$\ell_W(w) = \min \left\{ \ell(w') \,\middle|\, w' \in S^* \text{ and } w = w' \right\}.$$

So for example, if $s \in S$, then $\ell(ss) = 2$ while $\ell_W(ss) = 0$. If $w \in S^*$ and $\ell(w) = \ell_W(w)$, we say that w is *reduced*. If $w \in S^*$ is reduced and $s \in S$, then either ws (resp. sw) is reduced, or $w = w's$ (resp. $w = sw'$) for some reduced $w' \in S^*$ with $\ell(w') = \ell(w) - 1$. If $w \equiv s_1 \cdots s_k \in S^*$ is reduced, we define $\mathbf{w} = \sigma_{s_1} \cdots \sigma_{s_k} \in \mathcal{B}_\Gamma$. If $w, w' \in S^*$ are reduced and $w = w'$, then $\mathbf{w} = \mathbf{w}'$.

For $w \in S^*$, $s \in S$ with ws reduced, define

$$\begin{aligned} a_{w,s} &= \mathbf{w}\sigma_s^2\mathbf{w}^{-1} \quad \in \mathcal{P}_\Gamma \\ y_{w,s} &= \mathbf{w}\sigma_s\tau_s\mathbf{w}^{-1} \in \mathcal{SP}_\Gamma, \end{aligned}$$

and put

$$\begin{aligned} A &= \{a_{w,s} \,|\, w \in S^*\,,\ s \in S\,,\ ws \text{ is reduced}\} \\ Y &= \{y_{w,s} \,|\, w \in S^*\,,\ s \in S\,,\ ws \text{ is reduced}\}. \end{aligned}$$

Proposition 1 ([6], Corollary 6) *The pure Artin monoid $\mathcal{P}_\Gamma$ is generated by A.* □

The $a_{w,s}$ are calculated explicitly in [6] for some types Γ.

Lemma 2 *Let $t \in S$ and $y \in Y$. Then there exists $y' \in Y$ and $p \in \mathcal{P}_\Gamma$ such that*

$$\sigma_t y \sigma_t^{-1} = p y' p^{-1}.$$

Proof Suppose $y = y_{w,s}$ where $w \in S^*$, $s \in S$, and ws is reduced. If tws is reduced, then

$$\sigma_t y \sigma_t^{-1} = (\sigma_t \mathbf{w}) \sigma_s \tau_s (\sigma_t \mathbf{w})^{-1} = y_{tw,s} \in Y$$

and we are done. So suppose tws is not reduced. Then $ws = tw'$ for some $w' \in S^*$ with $\ell(w') = \ell(ws) - 1$. Since both ws and tw' are reduced, we conclude that $\mathbf{w}\sigma_s = \sigma_t \mathbf{w}'$, and so $\mathbf{w} = \sigma_t \mathbf{w}' \sigma_s^{-1}$. Now

$$\begin{aligned}
\sigma_t y \sigma_t^{-1} &= \sigma_t \mathbf{w} \sigma_s \tau_s \mathbf{w}^{-1} \sigma_t^{-1} \\
&= \sigma_t (\sigma_t \mathbf{w}' \sigma_s^{-1}) \sigma_s \tau_s (\sigma_t \mathbf{w}' \sigma_s^{-1})^{-1} \sigma_t^{-1} \\
&= \sigma_t^2 \mathbf{w}' \sigma_s \tau_s (\mathbf{w}')^{-1} \sigma_t^{-2}.
\end{aligned}$$

If $w's$ is reduced, then we are done with $y' = y_{w',s} \in Y$ and $p = \sigma_t^2 \in \mathcal{P}_\Gamma$. So suppose $w's$ is not reduced. Then $w' = w''s$ for some $w'' \in S^*$ with $\ell(w'') = \ell(w') - 1$. Since both w' and $w''s$ are reduced, we conclude that $\mathbf{w}' = \mathbf{w}'' \sigma_s$. Thus

$$\begin{aligned}
\sigma_t y \sigma_t^{-1} &= \sigma_t^2 \mathbf{w}' \sigma_s \tau_s (\mathbf{w}')^{-1} \sigma_t^{-2} && \text{from above} \\
&= \sigma_t^2 (\mathbf{w}'' \sigma_s) \sigma_s \tau_s (\mathbf{w}'' \sigma_s)^{-1} \sigma_t^{-2} \\
&= \sigma_t^2 \mathbf{w}'' \sigma_s \tau_s (\mathbf{w}'')^{-1} \sigma_t^{-2},
\end{aligned}$$

and we are done with $y' = y_{w'',s} \in Y$ and $p = \sigma_t^2 \in \mathcal{P}_\Gamma$. □

Corollary 3 *Let $\beta \in \mathcal{B}_\Gamma$ and $y \in Y$. Then there exists $y' \in Y$ and $p \in \mathcal{P}_\Gamma$ such that*

$$\beta y \beta^{-1} = p y' p^{-1}.$$

Proof Suppose $y \in Y$ and $\beta = \sigma_{s_1}^{\varepsilon_1} \cdots \sigma_{s_k}^{\varepsilon_k}$ where $s_1, \dots, s_k \in S$ and $\varepsilon_1, \dots, \varepsilon_k \in \{\pm 1\}$. We prove the result by induction on k. If $k = 0$ then the result is trivial. The case $k = 1$ and $\varepsilon_1 = 1$ is covered by Lemma 2. If $k = 1$ and $\varepsilon_1 = -1$, then by the same lemma we have

$$\sigma_{s_1}^{-1} y \sigma_{s_1} = \sigma_{s_1}^{-2} \sigma_{s_1} y \sigma_{s_1}^{-1} \sigma_{s_1}^2 = \sigma_{s_1}^{-2} p y' p^{-1} \sigma_{s_1}^2 = (\sigma_{s_1}^{-2} p) y' (\sigma_{s_1}^{-2} p)^{-1}$$

for some $y' \in Y$ and $p \in \mathcal{P}_\Gamma$, and we are done since $\sigma_{s_1}^{-2} p \in \mathcal{P}_\Gamma$. Suppose now that $k \geq 2$, and put $\beta' = \sigma_{s_1}^{\varepsilon_1} \cdots \sigma_{s_{k-1}}^{\varepsilon_{k-1}}$. Then by induction we have

$$\begin{aligned}
\beta y \beta^{-1} &= \beta' \sigma_{s_k}^{\varepsilon_k} y \sigma_{s_k}^{-\varepsilon_k} (\beta')^{-1} \\
&= \beta' p y' p^{-1} (\beta')^{-1} && \text{for some } y' \in Y,\ p \in \mathcal{P}_\Gamma \\
&= \big(\beta' p (\beta')^{-1}\big) \beta' y' (\beta')^{-1} \big(\beta' p^{-1} (\beta')^{-1}\big) \\
&= \big(\beta' p (\beta')^{-1}\big) p' y'' (p')^{-1} \big(\beta' p^{-1} (\beta')^{-1}\big) && \text{for some } y'' \in Y,\ p' \in \mathcal{P}_\Gamma \\
&= \big[\big(\beta' p (\beta')^{-1}\big) p'\big] y'' \big[\big(\beta' p (\beta')^{-1}\big) p'\big]^{-1}
\end{aligned}$$

and we are done since $\big(\beta' p (\beta')^{-1}\big) p' \in \mathcal{P}_\Gamma$. □

The following was mentioned in [10] for type A_n. Here we give a proof for arbitrary type Γ.

Proposition 4 *The pure singular Artin monoid $\mathcal{SP}_\Gamma$ is generated by $A \cup A^{-1} \cup Y$.*

Proof Let $Z = A \cup A^{-1} \cup Y$. It is clear that $Z \subseteq \mathcal{SP}_\Gamma$. Now let $\beta \in \mathcal{SP}_\Gamma$. To prove the proposition it suffices to prove by induction on $N(\beta)$ that $\beta \in \langle Z \rangle$. If $N(\beta) = 0$ then the result is true by Proposition 1. Otherwise, write $\beta = \beta_1 \tau_s \beta_2$ where $N(\beta_1) = 0$. By inserting $\sigma_s^{-1}\sigma_s$ after β_1 (if necessary), we may in fact assume that $\beta = \beta_1 \sigma_s \tau_s \beta_2$. Notice that $\pi(\beta_1\beta_2) = \pi(\beta) = 1$ and that $N(\beta_1\beta_2) < N(\beta)$ so $\beta_1\beta_2 \in \langle Z \rangle$ by an inductive hypothesis. Now

$$\beta = \beta_1 \sigma_s \tau_s \beta_1^{-1} \beta_1 \beta_2 = (\beta_1 y_{1,s} \beta_1^{-1})(\beta_1\beta_2).$$

By Corollary 3, $\beta_1 y_{1,s} \beta_1^{-1} = p y' p^{-1}$ for some $y' \in Y$, $p \in \mathcal{P}_\Gamma = \langle A \rangle$. Thus $\beta \in \langle Z \rangle$, completing the proof. □

A Coxeter graph Γ is said to have the FRZ property if for all $\beta \in \mathcal{SB}_\Gamma$ and $s, t \in S$, the following conditions are equivalent.

$$\beta\sigma_s = \sigma_t\beta \qquad \text{(FRZ1)}$$
$$\beta\sigma_s^m = \sigma_t^m\beta \qquad \text{for some } 0 \neq m \in \mathbb{Z} \qquad \text{(FRZ2)}$$
$$\beta\tau_s = \tau_t\beta \qquad \text{(FRZ3)}$$
$$\beta\tau_s^m = \tau_t^m\beta \qquad \text{for some } 0 \neq m \in \mathbb{N}. \qquad \text{(FRZ4)}$$

The property is named after Fenn, Rolfsen, and Zhu, who proved in [7] that the property holds in $\mathcal{SB}_n$. It is now known to be true when W_Γ is of FC type [8]. A Coxeter group is of FC type if every subset $X \subseteq S$ which satisfies $m_{st} \neq \infty$ for all $s, t \in X$ generates a finite subgroup of W_Γ. See also [1].

Proposition 5 *Suppose Γ is a Coxeter graph which has the FRZ property. Let $\beta_1, \beta_2 \in \mathcal{B}_\Gamma$, $s, t \in S$, and $w_s, w_t \in W_\Gamma$ such that $w_s s$ and $w_t t$ are reduced. Then the following statements are equivalent:*

$$\beta_1 a_{w_s,s} \beta_1^{-1} = \beta_2 a_{w_t,t} \beta_2^{-1} \qquad \text{(a)}$$
$$\beta_1 y_{w_s,s} \beta_1^{-1} = \beta_2 y_{w_t,t} \beta_2^{-1}. \qquad \text{(b)}$$

Proof If (a) holds then we have $\beta_1 \mathbf{w}_s \sigma_s^2 \mathbf{w}_s^{-1} \beta_1^{-1} = \beta_2 \mathbf{w}_t \sigma_t^2 \mathbf{w}_t^{-1} \beta_2^{-1}$. Rearranging gives

$$(\mathbf{w}_t^{-1}\beta_2^{-1}\beta_1\mathbf{w}_s)\sigma_s^2 = \sigma_t^2(\mathbf{w}_t^{-1}\beta_2^{-1}\beta_1\mathbf{w}_s).$$

By the FRZ property, we have

$$(\mathbf{w}_t^{-1}\beta_2^{-1}\beta_1\mathbf{w}_s)\sigma_s = \sigma_t(\mathbf{w}_t^{-1}\beta_2^{-1}\beta_1\mathbf{w}_s)$$
$$(\mathbf{w}_t^{-1}\beta_2^{-1}\beta_1\mathbf{w}_s)\tau_s = \tau_t(\mathbf{w}_t^{-1}\beta_2^{-1}\beta_1\mathbf{w}_s),$$

and so

$$(\mathbf{w}_t^{-1}\beta_2^{-1}\beta_1\mathbf{w}_s)\sigma_s\tau_s = \sigma_t\tau_t(\mathbf{w}_t^{-1}\beta_2^{-1}\beta_1\mathbf{w}_s).$$

Rearranging gives (b). The other implication is immediate from the ϕ map. □

From now on we will concentrate on the case $\Gamma = I_2(p)$. For simplicity we will write σ_i for σ_{s_i} and τ_i for τ_{s_i} $(i = 1, 2)$. We will also denote $\mathcal{B}_{I_2(p)}$ by $\mathcal{B}_I$, and similarly for $\mathcal{P}_I$, $\mathcal{SP}_I$, etc. So $\mathcal{SB}_I$ has a presentation with generators $\sigma_i^{\pm 1}, \tau_i$ $(i = 1, 2)$, and defining relations

$$\begin{aligned}
\sigma_i^{\pm 1}\sigma_i^{\mp 1} &= 1 && \text{for } i = 1, 2 \\
\sigma_i \tau_i &= \tau_i \sigma_i && \text{for } i = 1, 2 \\
\langle \sigma_1 \sigma_2 \rangle^p &= \langle \sigma_2 \sigma_1 \rangle^p && \\
\tau_1 \langle \sigma_2 \sigma_1 \rangle^{p-1} &= \langle \sigma_2 \sigma_1 \rangle^{p-1} \tau_{1\wedge 2} && \\
\tau_2 \langle \sigma_1 \sigma_2 \rangle^{p-1} &= \langle \sigma_1 \sigma_2 \rangle^{p-1} \tau_{2\wedge 1}, &&
\end{aligned}$$

where $i \wedge j = i$ if p is even or $i \wedge j = j$ if p is odd.

For $i \in \{1, \ldots, p-1\}$ let

$$\begin{aligned}
a_i &= \langle \sigma_1 \sigma_2 \rangle^{i-1} \sigma_d^2 \langle \sigma_1 \sigma_2 \rangle^{-(i-1)} \\
y_i &= \langle \sigma_1 \sigma_2 \rangle^{i-1} \sigma_d \tau_d \langle \sigma_1 \sigma_2 \rangle^{-(i-1)}
\end{aligned}$$

where $d = 1$ if i is odd or $d = 2$ if i is even. Also let

$$\begin{aligned}
b &= \sigma_2^2 \\
z &= \sigma_2 \tau_2.
\end{aligned}$$

It is shown in [6] that $A = \{a_1, \ldots, a_{p-1}, b\}$ and so $Y = \{y_1, \ldots, y_{p-1}, z\}$, and $\mathcal{SP}_I$ is generated by $A \cup A^{-1} \cup Y$ by Proposition 4. Let $\mathcal{U}_I$ be the subgroup of $\mathcal{P}_I$ generated by $a_1, \ldots, a_{p-1}$, and let $\mathcal{P}_2$ be the (infinite cyclic) subgroup generated by b. The following is from [6] where similar results are proved for other types.

Proposition 6 *$\mathcal{U}_I$ is freely generated by $a_1, \ldots, a_{p-1}$. Further, $\mathcal{P}_I = \mathcal{U}_I \rtimes \mathcal{P}_2$. The action of b on the a_i is given by*

$$b a_i b^{-1} = (a_{p-1} \cdots a_1)^{-1} a_i (a_{p-1} \cdots a_1) \qquad \textit{for} \quad i = 1, \ldots, p-1,$$

and these are defining relations for $\mathcal{P}_I$. □

For example, when $p = 3$, the classical three-string pure braid group $\mathcal{P}_3 = \mathcal{P}_{I_2(3)}$ has a presentation

$$\mathcal{P}_3 = \langle a_1, a_2, b \,|\, b a_1 b^{-1} = (a_2 a_1)^{-1} a_1 (a_2 a_1)\,,\ b a_2 b^{-1} = a_1^{-1} a_2 a_1 \rangle$$

in terms of generators $a_1 = \sigma_1^2$, $a_2 = \sigma_1 \sigma_2^2 \sigma_1^{-1}$, and $b = \sigma_2^2$.

Let $\mathcal{V}_I$ be the submonoid of $\mathcal{SP}_I$ generated by $a_i^{\pm 1}, y_i$ $(i = 1, \ldots, p-1)$. By Propositions 5 and 6, we have

$$b y_i b^{-1} = (a_{p-1} \cdots a_1)^{-1} y_i (a_{p-1} \cdots a_1) \qquad \text{for} \quad i = 1, \ldots, p-1.$$

Thus, if w is a word in the generators of $\mathcal{SP}_I$ with no occurrence of the letter z, then we may write $w = vp$ where $v \in \mathcal{V}_I$ and $p \in \mathcal{P}_2$. To see that this expression is unique (up to equivalence of words), suppose that $w = v'p'$ is also such an expression. Then $\phi(v)p = \phi(w) = \phi(v')p'$. Since $\phi(v), \phi(v') \in \mathcal{U}_I$ and $p, p' \in \mathcal{P}_2$, we see that $p = p'$, and consequently $v = v'$.

Let $\beta \in \mathcal{V}_I$. Since a_i and y_i commute for each i, we may assume that in β, no $a_i^{\pm 1}$ immediately precedes a y_i. Thus β may be written as a product of words of the form a_i^{ℓ} (with $0 \neq \ell \in \mathbb{Z}$), and $y_i^m a_i^n$ (with $1 \leq m \in \mathbb{Z}$ and $n \in \mathbb{Z}$). We call these words *simple words*. We may also assume that the subscripts appearing in adjacent simple words are different. As in [10], we call this a *freely reduced form* of β, which we will shortly see is unique.

3 The Proof of the Conjecture for Type $I_2(p)$

We begin this section by stating some results which are proved in [10]. There they are proved for type A_n, but the proofs work unmodified for any Γ.

Define a homomorphism $\psi : \mathcal{SB}_\Gamma \to \mathbb{Z}\mathcal{B}_\Gamma$ by $\psi(\sigma_i^{\pm 1}) = \sigma_i^{\pm 1}$ and $\psi(\tau_i) = \sigma_i + \sigma_i^{-1}$.

Lemma 7 *(i) If $\beta, \gamma \in \mathcal{SB}_\Gamma$, then $\eta(\beta) = \eta(\gamma) \iff \psi(\beta) = \psi(\gamma)$. Thus η is injective if and only if ψ is injective.*

(ii) The image of $\mathcal{SB}_\Gamma$ under ψ contains no zero divisors. Thus, if $\beta, \gamma, \delta \in \mathcal{SB}_\Gamma$, then $\psi(\beta\delta) = \psi(\gamma\delta) \iff \psi(\beta) = \psi(\gamma) \iff \psi(\delta\beta) = \psi(\delta\gamma)$. $\square$

Lemma 8 *The map ψ is injective if and only if the restriction of ψ to $\mathcal{SP}_\Gamma$ is injective.*

Proof Suppose $\psi|_{\mathcal{SP}_\Gamma}$ is injective, and take $\beta, \gamma \in \mathcal{SB}_\Gamma$ with $\psi(\beta) = \psi(\gamma)$. By looking at the nonzero terms in $\psi(\beta) = \sum_{x \in \mathcal{B}_\Gamma} c_x x = \psi(\gamma)$, we see that $\pi(\beta) = \pi(\gamma)$. Choose $\delta \in \mathcal{B}_\Gamma$ such that $\pi(\delta) = \pi(\beta)$. Then $\psi(\beta\delta^{-1}) = \psi(\gamma\delta^{-1})$. But $\beta\delta^{-1}, \gamma\delta^{-1} \in \mathcal{SP}_\Gamma$, so that $\beta\delta^{-1} = \gamma\delta^{-1}$ and $\beta = \gamma$. $\square$

The following was proved as part of Theorem 3 in [10].

Theorem 9 *Let $C = \{c_1, \ldots, c_n\}$ and $D = \{d_1, \ldots, d_n\}$. Denote by $F(C)$ the free group (freely) generated by C. Suppose that M is a monoid generated by $C \cup C^{-1} \cup D$ and that $c_i d_i = d_i c_i$ in M for all i. Suppose also that there is a well defined monoid homomorphism $\xi : M \to \mathbb{Z}F(C)$ such that $\xi(c_i^{\pm 1}) = c_i^{\pm 1}$ and $\xi(d_i) = c_i + 1$. Then ξ is injective.* $\square$

We again restrict our attention to $\Gamma = I_2(p)$. Notice that $\psi(a_i^{\pm 1}) = a_i^{\pm 1}$ and that $\psi(y_i) = a_i + 1$. So $\mathcal{V}_I$ is an example of a monoid satisfying the assumptions of the theorem. The proof of Theorem 9 given in [10] makes use of a freely reduced form for elements of M which is defined analogously to that defined above for $\mathcal{V}_I$. There it is shown that the freely reduced form of $m \in M$ may be constructed simply by knowing $\xi(m) = \sum_{x \in F(C)} c_x x$.

Corollary 10 *The restriction of ψ to $\mathcal{V}_I$ is injective. Further, the freely reduced form of an element of $\mathcal{V}_I$ is unique.* $\square$

So far we know that if $\psi(\beta) = \psi(\gamma)$ where $\beta, \gamma \in \mathcal{SP}_I$ do not involve b or z, then $\beta = \gamma$. We now allow b to appear, but not z.

Corollary 11 *If $\psi(\beta) = \psi(\gamma)$ where $\beta, \gamma \in \mathcal{SP}_I$ have no appearances of z, then $\beta = \gamma$.*

Proof Suppose $\psi(\beta) = \psi(\gamma)$ where $\beta = v_1 p_1$ and $\gamma = v_2 p_2$ with $v_1, v_2 \in \mathcal{V}_I$ and $p_1, p_2 \in \mathcal{P}_2$. Since $\phi(\beta)$ and $\phi(\gamma)$ are the unique monomials of maximal exponent sum in $\psi(\beta)$ and $\psi(\gamma)$ respectively, we see that $\phi(\beta) = \phi(\gamma)$. That is, $\phi(v_1)p_1 = \phi(v_2)p_2$. As before, we conclude that $p_1 = p_2$. But then $\psi(v_1) = \psi(\beta p_1^{-1}) = \psi(\gamma p_1^{-1}) = \psi(v_2)$. By Corollary 10, $v_1 = v_2$ and so $\beta = \gamma$. $\square$

We now proceed to words which may involve z, but not as their first singular generators. Define $\varepsilon(b) = 1$ and $\varepsilon(a_i) = 0$ for $i = 1, \ldots, p-1$. As a consequence of the defining relations in Proposition 6, ε extends to a well defined homomorphism.

Lemma 12 *Suppose that $\beta, \gamma \in \mathcal{SP}_I$ with $\psi(\beta) = \psi(\gamma)$. Suppose also that $N(\beta) = N(\gamma) \geq 1$ and that the first singular generator in a word over $A \cup A^{-1} \cup Y$ representing β (resp. γ) is y_i (resp. y_j). Then $i = j$ and there exists $\alpha \in \mathcal{SP}_I$ with $N(\alpha) = 1$, and $\beta', \gamma' \in \mathcal{SP}_I$ such that $\beta = \alpha\beta'$ and $\gamma = \alpha\gamma'$.*

Proof Write

$$\beta = \beta_0 y_i \beta', \tag{1}$$

$$\gamma = \gamma_0 y_j \gamma', \tag{2}$$

where $\beta_0, \gamma_0 \in \mathcal{P}_I$ and $\beta', \gamma' \in \mathcal{SP}_I$. By Proposition 6 and the comments following it, we may assume that $\beta_0, \gamma_0 \in \mathcal{U}_I$. We may also assume that any power of a_i (resp. a_j) at the end of β_0 (resp. γ_0) has been absorbed by β' (resp. γ'). Replace all generators z appearing in β' and γ' by b, and denote the resulting words by β'_+ and γ'_+. Also, let $\beta_+ = \beta_0 y_i \beta'_+$ and $\gamma_+ = \gamma_0 y_j \gamma'_+$.

Write $\psi(\beta) = \sum_{x \in \mathcal{P}_I} c_x x$. We group terms according to their image under ε to obtain

$$\psi(\beta) = \sum_{r \in \mathbb{Z}} \Big(\sum_{\substack{x \in \mathcal{P}_I \\ \varepsilon(x) = r}} c_x x \Big).$$

The inner sum corresponding to the maximal value of r is precisely $\psi(\beta_+)$. Since $\psi(\beta) = \psi(\gamma)$, we see that this sum is also $\psi(\gamma_+)$ so that $\psi(\beta_+) = \psi(\gamma_+)$. This implies that $\beta_+ = \gamma_+$ by Corollary 11. Now

$$\beta_0 y_i \beta'_+ = \beta_+ = \gamma_+ = \gamma_0 y_j \gamma'_+.$$

Since both sides do not contain any generator z, we may write

$$\beta'_+ = \beta''_+ p_1$$
$$\gamma'_+ = \gamma''_+ p_2,$$

where $\beta''_+, \gamma''_+ \in \mathcal{V}_I$ are in freely reduced form, and $p_1, p_2 \in \mathcal{P}_2$. So

$$\beta_0 y_i \beta''_+ p_1 = \gamma_0 y_j \gamma''_+ p_2.$$

As before, $p_1 = p_2$ and $\beta_0 y_i \beta''_+ = \gamma_0 y_j \gamma''_+$. By the uniqueness of the freely reduced form, $i = j$ and $\beta_0 = \gamma_0$ since y_i and y_j are the first singular generators appearing on both sides. Now put $\alpha = \beta_0 y_i$. Then equations (1) and (2) give the statement in the lemma. □

Lemma 13 *If $j \in \{1, \ldots, p-1\}$, then $\sigma_2 y_j \sigma_2^{-1} = (a_{p-j-1} \cdots a_1)^{-1} y_{p-j} (a_{p-j-1} \cdots a_1)$.*

Proof To see this it is most convenient to consider separate cases depending on the parity of p and j. Here we treat the case when p and j are both even, the others being similar. Now

$$\begin{aligned}
\sigma_2 y_j \sigma_2^{-1} &= \sigma_2 \langle \sigma_1 \sigma_2 \rangle^{j-1} \sigma_2 \tau_2 \langle \sigma_1 \sigma_2 \rangle^{-(j-1)} \sigma_2^{-1} \\
&= \langle \sigma_2 \sigma_1 \rangle^{j} \sigma_2 \tau_2 \langle \sigma_2 \sigma_1 \rangle^{-j} \\
&= \langle \sigma_1 \sigma_2 \rangle^{-(p-j-1)} \langle \sigma_1 \sigma_2 \rangle^{p-j-1} \langle \sigma_2 \sigma_1 \rangle^{j} \sigma_2 \tau_2 \langle \sigma_2 \sigma_1 \rangle^{-j} \langle \sigma_1 \sigma_2 \rangle^{-(p-j-1)} \langle \sigma_1 \sigma_2 \rangle^{p-j-1} \\
&= \langle \sigma_1 \sigma_2 \rangle^{-(p-j-1)} \langle \sigma_1 \sigma_2 \rangle^{p} \tau_2 \langle \sigma_1 \sigma_2 \rangle^{-(p-1)} \langle \sigma_1 \sigma_2 \rangle^{p-j-1} \\
&= \langle \sigma_1 \sigma_2 \rangle^{-(p-j-1)} \langle \sigma_2 \sigma_1 \rangle^{p} \langle \sigma_1 \sigma_2 \rangle^{-(p-1)} \tau_2 \langle \sigma_1 \sigma_2 \rangle^{p-j-1} \\
&= \langle \sigma_1 \sigma_2 \rangle^{-(p-j-1)} \sigma_2 \tau_2 \langle \sigma_1 \sigma_2 \rangle^{p-j-1} \\
&= \langle \sigma_1 \sigma_2 \rangle^{-(p-j-1)} \langle \sigma_1 \sigma_2 \rangle^{-(p-j-1)} \langle \sigma_1 \sigma_2 \rangle^{p-j-1} \sigma_2 \tau_2 \langle \sigma_1 \sigma_2 \rangle^{-(p-j-1)} \langle \sigma_1 \sigma_2 \rangle^{p-j-1} \langle \sigma_1 \sigma_2 \rangle^{p-j-1} \\
&= \big(\langle \sigma_1 \sigma_2 \rangle^{p-j-1} \big)^{-2} y_{p-j} \big(\langle \sigma_1 \sigma_2 \rangle^{p-j-1} \big)^{2}.
\end{aligned}$$

We now prove by induction that

$$\big(\langle\sigma_1\sigma_2\rangle^k\big)^2 = a_k\cdots a_1 \tag{$*$}$$

for any odd integer $1 \le k \le p-1$. This is clear when $k=1$. Suppose $(*)$ holds for some odd integer $1 \le k \le p-3$. Then

$$\begin{aligned}
a_{k+2}a_{k+1}a_k\cdots a_1 &= \langle\sigma_1\sigma_2\rangle^{k+1}\sigma_1^2\langle\sigma_1\sigma_2\rangle^{-(k+1)}\langle\sigma_1\sigma_2\rangle^k\sigma_2^2\langle\sigma_1\sigma_2\rangle^{-k}\big(\langle\sigma_1\sigma_2\rangle^k\big)^2\\
&= \langle\sigma_1\sigma_2\rangle^{k+1}\sigma_1^2\sigma_2\langle\sigma_1\sigma_2\rangle^k\\
&= \big(\langle\sigma_1\sigma_2\rangle^{k+2}\big)^2.
\end{aligned}$$

The reader is invited to consider the three other cases. □

We are now ready to prove our main result.

Theorem 14 *The map $\psi : \mathcal{SB}_I \to \mathbb{Z}\mathcal{B}_I$ is injective.*

Proof Suppose that $\beta, \gamma \in \mathcal{SP}_I$ with $\psi(\beta) = \psi(\gamma)$ but $\beta \neq \gamma$. Suppose also that the number $N = N(\beta)$ is minimal among all such counterexamples. In particular we must have $N \ge 1$. Write

$$\begin{aligned}
\beta &= \beta_1 x \beta_2\\
\gamma &= \gamma_1 x' \gamma_2
\end{aligned}$$

where $\beta_1, \gamma_1 \in \mathcal{P}_I$, $\beta_2, \gamma_2 \in \mathcal{SP}_I$, and x and x' are the first singular generators appearing in β and γ respectively. We break the proof up into a number of cases.

Case 1 Suppose $x = y_i$ and $x' = y_j$ for some i, j. We apply Lemma 12. Write $\beta = \alpha\beta'$ and $\gamma = \alpha\gamma'$ in the notation of the lemma. Then by Lemma 7 Part (ii), $\psi(\beta') = \psi(\gamma')$. But $\beta' \neq \gamma'$, contradicting the minimality of N.

Case 2 Suppose $x = x' = z$. Note that $\sigma_1 z \sigma_1^{-1} = y_2$. Now $\psi(\sigma_1\beta\sigma_1^{-1}) = \psi(\sigma_1\gamma\sigma_1^{-1})$, and

$$\sigma_1\beta\sigma_1^{-1} = (\sigma_1\beta_1\sigma_1^{-1})(\sigma_1 z\sigma_1^{-1})(\sigma_1\beta_2\sigma_1^{-1}) = \beta_1' y_2 \beta_2',$$

where $\beta_1' = \sigma_1\beta_1\sigma_1^{-1} \in \mathcal{P}_I$ and $\beta_2' = \sigma_1\beta_2\sigma_1^{-1} \in \mathcal{SP}_I$. Similarly, $\sigma_1\gamma\sigma_1^{-1} = \gamma_1' y_2 \gamma_2'$ with $\gamma_1' \in \mathcal{P}_I$ and $\gamma_2' \in \mathcal{SP}_I$. By Case 1 we have $\sigma_1\beta\sigma_1^{-1} = \sigma_1\gamma\sigma_1^{-1}$ and so $\beta = \gamma$, a contradiction.

Case 3 We now consider the case in which $x = z$ and $x' = y_i$ for some $i \neq 2$. If $i = 1$, then note that $\sigma_1 y_1 \sigma_1^{-1} = y_1$ and, as in Case 2, we see that

$$\begin{aligned}
\sigma_1\beta\sigma_1^{-1} &= \beta_1' y_2 \beta_2'\\
\sigma_1\gamma\sigma_1^{-1} &= \gamma_1' y_1 \gamma_2',
\end{aligned}$$

and we are done by Case 1. Suppose $i \ge 3$. Then, again using the notation $d = 1$ if i is odd, while $d = 2$ if i is even, we have

$$\begin{aligned}
\sigma_1 y_i \sigma_1^{-1} &= \sigma_1\langle\sigma_1\sigma_2\rangle^{i-1}\sigma_d\tau_d\langle\sigma_1\sigma_2\rangle^{-(i-1)}\sigma_1^{-1}\\
&= \sigma_1^2\sigma_2\langle\sigma_1\sigma_2\rangle^{i-3}\sigma_d\tau_d\langle\sigma_1\sigma_2\rangle^{-(i-3)}\sigma_2^{-1}\sigma_1^{-2}\\
&= a_1\sigma_2 y_{i-2}\sigma_2^{-1}a_1^{-1}\\
&= a_1(a_{p-i+1}\cdots a_1)^{-1}y_{p-i+2}(a_{p-i+1}\cdots a_1)a_1^{-1} && \text{by Lemma 13.}
\end{aligned}$$

Thus, with notation as in Case 2,

$$\begin{aligned}\sigma_1\beta\sigma_1^{-1} &= \beta_1' y_2 \beta_2' \\ \sigma_1\gamma\sigma_1^{-1} &= \gamma_1'(a_{p-i-1}\cdots a_2)y_{p-i+2}(a_{p-i+1}\cdots a_2)\gamma_2',\end{aligned}$$

and we are done again by Case 1.

Case 4 Finally we consider the case in which $x = z$ and $x' = y_2$. Let κ be the automorphism of the graph $I_2(p)$ which switches the two vertices. Then κ induces automorphisms of $\mathbb{Z}\mathcal{B}_I$ and $\mathcal{SB}_I$ both of which we denote simply by κ, and the following diagram commutes:

$$\begin{tikzcd} \mathcal{SB}_I \arrow[r, "\kappa"] \arrow[d, "\psi"'] & \mathcal{SB}_I \arrow[d, "\psi"] \\ \mathbb{Z}\mathcal{B}_I \arrow[r, "\kappa"'] & \mathbb{Z}\mathcal{B}_I \end{tikzcd}$$

So $\psi(\kappa(\beta)) = \kappa(\psi(\beta)) = \kappa(\psi(\gamma)) = \psi(\kappa(\gamma))$. Notice that $\kappa(z) = y_1$. But then $\kappa(\beta) = \kappa(\beta_1)y_1\kappa(\beta_2)$, and so the first singular generator of $\kappa(\beta)$ is y_1. Similarly, $\kappa(y_2) = \sigma_2 y_1 \sigma_2^{-1} = (a_{p-2}\cdots a_1)^{-1}y_{p-1}(a_{p-2}\cdots a_1)$, so the first singular generator in $\kappa(\gamma)$ is y_{p-1}. By Case 1, we have $\kappa(\beta) = \kappa(\gamma)$ and $\beta = \gamma$. $\square$

Finally we remark that Case 4 may *almost* be proved by reduction to Case 3. Notice that $\sigma_1 z \sigma_1^{-1} = y_2$ and $\sigma_1 y_2 \sigma_1^{-1} = a_1 z a_1^{-1}$ so conjugation by σ_1 does not help. However $\sigma_2 z \sigma_2^{-1} = z$, while $\sigma_2 y_2 \sigma_2^{-1} = (a_{p-3}\cdots a_1)^{-1}y_{p-2}(a_{p-3}\cdots a_1)$ by Lemma 13, and we may reduce to Case 3 unless $p - 2 = 2$ (ie. $p = 4$).

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